

**POISSON INVOLUTIONS, SPIN CALOGERO-MOSER
SYSTEMS ASSOCIATED WITH SYMMETRIC LIE
SUBALGEBRAS AND THE SYMMETRIC SPACE
SPIN RUIJSENAARS-SCHNEIDER MODELS**

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ABSTRACT. We develop a general scheme to construct integrable systems starting from realizations in symmetric coboundary dynamical Lie algebroids and symmetric coboundary Poisson groupoids. The method is based on the successive use of Dirac reduction and Poisson reduction. Then we show that certain spin Calogero-Moser systems associated with symmetric Lie subalgebras can be studied in this fashion. We also consider some spin-generalized Ruijsenaars-Schneider equations which correspond to the N -soliton solutions of $A_n^{(1)}$ affine Toda field theory. In this case, we show how the equations are obtained from the Dirac reduction of some Hamiltonian system on a symmetric coboundary dynamical Poisson groupoid.

1. Introduction.

In the last few years, a groupoid-theoretic scheme based on the coboundary dynamical Poisson groupoids and their corresponding Lie bialgebroids was introduced in the study of certain integrable Hamiltonian systems and their solutions [LX],[L1],[L2],[L3]. As is well-known, these geometric objects are naturally associated with so-called classical dynamical r-matrices [EV],[BKS] which first appeared in the context of Wess-Zumino-Witten (WZW) conformal field theory [BDF],[F].

In this paper, we shall continue to use these geometric objects to unify the study of a variety of Hamiltonian systems known under the general name of spin Calogero-Moser systems and spin Ruijsenaars-Schneider models. More specifically, we shall consider in this work examples of such systems which turn out to be realizable in the stable loci of the geometric objects mentioned above under Poisson involutions. As we know through the work in [X], the stable locus of a Poisson involution is an example of a class of submanifolds with induced Poisson structures which the author in [X] called Dirac submanifolds. (The fact that the stable locus of a Poisson involution carries a natural induced Poisson structure was also noted by the authors

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in [FV].) Indeed, as we shall explain below, it is advantageous to formulate several of our results in this broader framework.

We now give an outline of our approach. As starting point, we consider certain spin Calogero-Moser systems (resp. spin Ruijsenaars-Schneider models) which can be realized in the dual bundles of symmetric coboundary dynamical Lie algebroids (resp. symmetric coboundary dynamical Poisson groupoids). For these Hamiltonian systems, the underlying Poisson manifolds as well as their realization spaces are both Hamiltonian H -spaces which carry natural Poisson involutions. The construction of the integrable systems of interests then proceeds in two stages. In the first stage, we apply Dirac reduction (which will be developed here) to reduce the initial realization maps to ones between the stable loci of Poisson involutions. In this way, we obtain the realization of the Dirac reduction of the afore-mentioned systems. In general, these reduced systems are not integrable systems (see Theorem 3.15, Theorem 3.18 and Section 4.2 for exceptions). However, as it turns out, the stable loci are Hamiltonian D -spaces for some subgroup D of H . Moreover, the natural invariant functions Poisson commute on certain fibers of the equivariant momentum maps. Consequently, we can apply Poisson reduction (and this is the second stage) to obtain the associated integrable systems.

There are several motivations for this work. One of these has come from the desire to understand the Hamiltonian formulation as well as the integrability of the equations of motion which arise from the so-called level dynamics approach in random matrix theory [Y],[HKS],[NM],[GRMN]. This connection is reflected in our choice of examples in Section 4. On the other hand, there is a well-known correspondence between the N -soliton solutions of the $A_n^{(1)}$ affine Toda field theory and some spin-generalized Ruijsenaars-Schneider equations [BH]. (Some of the variables in these equations actually depend on the choice of eigenvectors of a certain skew-Hermitian matrix V .) However, the Hamiltonian formulation of these equations has remained open. We shall give a solution to this problem in Section 5 below. As the reader will see, these equations are related to a symmetric coboundary dynamical Poisson groupoid (Γ, Σ) , where Γ is associated to a hyperbolic dynamical r -matrix, and Σ is a Poisson involution on Γ . More precisely, they can be obtained from a Hamiltonian system on the stable locus Γ^Σ of Σ by restricting the equations of motion to an appropriate fiber of the momentum map. Consequently, the system which is invariant under the gauge freedom in picking the eigenvectors of V is an integrable Hamiltonian system on a Poisson reduction of Γ^Σ . Finally we remark that in the process of assembling the necessary machinery in order to tackle the

above problems, we will give the explicit expression for the Poisson structure on the stable locus of a Poisson involution on a coboundary dynamical Lie algebroid (resp. coboundary dynamical Poisson groupoid). Thus this answers a question raised in [X].

The paper is organized as follows. In section 2, we begin by recalling some basic facts about coboundary dynamical Lie algebroids and coboundary dynamical Poisson groupoids which will be used throughout the paper. In particular, we will discuss a subclass of such Lie algebroids defined by so-called classical dynamical r -matrices with spectral parameter. We will also recall what we mean by spin Calogero-Moser systems associated with this subclass of coboundary dynamical Lie algebroids. In section 3, the main goal is to develop a general scheme of constructing integrable systems based on realization in symmetric coboundary dynamical Poisson groupoids and the dual bundles of symmetric coboundary dynamical Lie algebroids. As we already mentioned above in the context of specific examples, the construction proceeds in two stages. For Dirac reduction, our main tool comes from an elementary result which shows how to reduce a Poisson map between two Poisson manifolds to one between their respective Dirac submanifolds (Theorem 3.2 and Corollary 3.5). From this, we also obtain a condition under which a Dirac submanifold Q of a Hamiltonian G -space P is Hamiltonian H -space for some Lie subgroup H of G (Proposition 3.6 and Corollary 3.7). There are two reasons for formulating our results in terms of Dirac submanifolds. First, the notion offers a better conceptual framework. Secondly, when formulated in this broader framework, the results are also applicable to the cosymplectic submanifolds [W1] (when P is symplectic, the cosymplectic submanifolds of P are precisely its symplectic submanifolds). In the special case when the Dirac submanifold is given by the stable locus of a Poisson involution on the dual bundle of a coboundary dynamical Lie algebroid (resp. coboundary dynamical Poisson groupoid), we also derive the intrinsic expression for the induced Poisson structure which is essential for our purpose here. In Section 4, we introduce several examples of spin Calogero-Moser systems associated with real symmetric Lie algebras. Then we show how the reduction procedure developed in Section 3 can be carried out to obtain the associated integrable systems of interests. In the special case when the Lie algebra \mathfrak{g} is $gl(N, \mathbb{C})$, we also provide a sketch of the Liouville integrability of the associated integrable models. Note that our goal of this section is suggestive rather than exhaustive in the sense that we have made no attempt to give a classification of systems which can be treated by our method. Finally, in Section 5, we consider the spin Ruijsenaars-Schneider mod-

els associated with a symmetric coboundary dynamical Poisson groupoid (Γ, Σ) . In this case, the realization map is just the identity map and it is easy to show how the scheme in Section 3 can be implemented. As we mentioned earlier, our goal here is explain how the spin-generalized Ruijsenaars-Schneider equations in [BH] are obtained from an invariant Hamiltonian system on Γ^Σ which is a special case of what we call symmetric space Ruijsenaars-Schneider models here.

To close, we remark that a factorization theory also exists for the solution of the Hamiltonian systems treated here (provided the classical dynamical r-matrix satisfies the modified dynamical Yang-Baxter equation), as is clear from assumptions A5 and G5 in Section 3 and the development in [L1],[L2]. For this reason, we do not give any details here.

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2. Preliminaries.

The purpose of this section is to recall some basic results about coboundary dynamical Lie algebroids and coboundary dynamical Poisson groupoids. For our applications in this work, we will pay special attention to a subclass of such Lie algebroids which are associated with so-called classical dynamical r-matrices with spectral parameter. We will also recall what we mean by spin Calogero-Moser systems associated with this subclass of coboundary dynamical Lie algebroids.

Let G be a connected Lie group, and $H \subset G$ a connected Lie subgroup. We shall denote by \mathfrak{g} and \mathfrak{h} the corresponding Lie algebras and let $\iota : \mathfrak{h} \longrightarrow \mathfrak{g}$ be the Lie inclusion. In what follows, the Lie groups and Lie algebras can be real or complex unless we specify otherwise.

We begin by recalling a fundamental construction in [EV] which gives a geometric interpretation of dynamical r-matrices in terms of Poisson groupoids. We shall, however, follow the formulation in [L1] and in particular we shall give the explicit expression for the Poisson structure which is essential for our purpose here. Let $U \subset \mathfrak{h}^*$ be a connected Ad_H^* -invariant open subset, we say that a smooth (resp. holomorphic) map $R : U \longrightarrow L(\mathfrak{g}^*, \mathfrak{g})$ (here and henceforth we denote by $L(\mathfrak{g}^*, \mathfrak{g})$ the set of linear maps from \mathfrak{g}^* to \mathfrak{g}) is a classical dynamical r-matrix if and only if it is pointwise skew-symmetric:

$$\langle R(q)(A), B \rangle = - \langle A, R(q)B \rangle \quad (2.1)$$

and satisfies the classical dynamical Yang-Baxter condition

$$\begin{aligned} & [R(q)A, R(q)B] + R(q)(ad_{R(q)A}^* B - ad_{R(q)B}^* A) \\ & + dR(q)\iota^* A(B) - dR(q)\iota^* B(A) + \langle dR(q)(\cdot)(A), B \rangle = \chi(A, B), \end{aligned} \quad (2.2)$$

where $\langle dR(q)(\cdot)(A), B \rangle$ is the element in \mathfrak{h} whose pairing with $\lambda \in \mathfrak{h}^*$ is given by $\langle dR(q)(\lambda)(A), B \rangle$ and $\chi : \mathfrak{g}^* \times \mathfrak{g}^* \longrightarrow \mathfrak{g}$ is G -equivariant, that is,

$$\chi(Ad_{g^{-1}}^* A, Ad_{g^{-1}}^* B) = Ad_g \chi(A, B) \quad (2.3)$$

for all $A, B \in \mathfrak{g}^*$, $g \in G$, and all $q \in U$.

The dynamical r -matrix is said to be H -equivariant if and only if

$$R(Ad_h^* q) = Ad_h \circ R(q) \circ Ad_h^* \quad (2.4)$$

for all $h \in H, q \in U$.

We shall equip $\Gamma = U \times G \times U$ with the trivial Lie groupoid structure over U [M] with target and source maps

$$\alpha(u, g, v) = u, \quad \beta(u, g, v) = v \quad (2.5)$$

and multiplication map

$$m((u, g, v), (v, g', w)) = (u, gg', w). \quad (2.6)$$

For a smooth (resp. holomorphic) function φ on Γ , we define its partial derivatives and its left and right gradients (with respect to G) by

$$\begin{aligned} \langle \delta_1 \varphi, u' \rangle &= \left. \frac{d}{dt} \right|_{t=0} \varphi(u + tu', g, v), \quad \langle \delta_2 \varphi, v' \rangle = \left. \frac{d}{dt} \right|_{t=0} \varphi(u, g, v + tv'), \quad u', v' \in \mathfrak{h}^* \\ \langle D\varphi, X \rangle &= \left. \frac{d}{dt} \right|_{t=0} \varphi(u, e^{tX} g, v), \quad \langle D'\varphi, X \rangle = \left. \frac{d}{dt} \right|_{t=0} \varphi(u, g e^{tX}, v), \quad X \in \mathfrak{g}. \end{aligned}$$

Theorem 2.1. (a) *The bracket*

$$\begin{aligned} \{\varphi, \psi\}_R(u, g, v) &= \langle u, [\delta_1 \varphi, \delta_1 \psi] \rangle - \langle v, [\delta_2 \varphi, \delta_2 \psi] \rangle \\ &\quad - \langle \iota \delta_1 \varphi, D\psi \rangle - \langle \iota \delta_2 \varphi, D'\psi \rangle \\ &\quad + \langle \iota \delta_1 \psi, D\varphi \rangle + \langle \iota \delta_2 \psi, D'\varphi \rangle \\ &\quad + \langle R(v)D'\varphi, D'\psi \rangle - \langle R(u)D\varphi, D\psi \rangle \end{aligned} \quad (2.7)$$

defines a Poisson structure on Γ if and only if $R : U \longrightarrow L(\mathfrak{g}^*, \mathfrak{g})$ is an H -equivariant classical dynamical r -matrix.

(b) *The trivial Lie groupoid Γ equipped with the Poisson bracket $\{\cdot, \cdot\}_R$ is a Poisson groupoid. Moreover, it is a Hamiltonian H -space under the natural left and right H -actions with equivariant momentum maps given by α and β respectively.*

We shall call the pair $(\Gamma, \{\cdot, \cdot\}_R)$ the *coboundary dynamical Poisson groupoid* associated to R . Note that the explicit expression for $\{\varphi, \psi\}_R$ in Theorem 2.1(a) above can be derived from the characterizing properties in [EV] and the corresponding expression for the general dynamical case can be found in [LP].

Let $A\Gamma = \bigcup_{q \in U} \{0_q\} \times \mathfrak{g} \times \mathfrak{h}^* \simeq TU \times \mathfrak{g}$ be the Lie algebroid of the trivial Lie groupoid Γ . (See [CdSW], [M] for details.) Then associated with $(\Gamma, \{\cdot, \cdot\}_R)$ is a Lie algebroid structure on the dual bundle $A^*\Gamma = \bigcup_{q \in U} \{0_q\} \times \mathfrak{g}^* \times \mathfrak{h} \simeq T^*U \times \mathfrak{g}^*$ as a consequence of Weinstein's coisotropic calculus [W2]. (See [LP] for a more general discussion and [BKS] for a different approach.) The anchor map $a_* : A^*\Gamma \longrightarrow TU$ of $A^*\Gamma$ of this Lie algebroid is given by

$$a_*(0_q, A, Z) = (q, \iota^* A - ad_Z^* q) \quad (2.8)$$

while the bracket $[\cdot, \cdot]_{A^*\Gamma}$ on $Sect(U, A^*\Gamma)$ has the following form [BKS], [L2]:

$$\begin{aligned} & [(0, \xi, Z), (0, \xi', Z')]_{A^*\Gamma}(q) \\ &= (0_q, d\xi'(q)(\iota^* \xi(q) - ad_{Z(q)}^* q) - d\xi(q)(\iota^* \xi'(q) - ad_{Z'(q)}^* q) \\ &\quad - ad_{R(q)\xi(q) - Z(q)}^* \xi'(q) + ad_{R(q)\xi'(q) - Z'(q)}^* \xi(q), \\ &\quad dZ'(q)(\iota^* \xi(q) - ad_{Z(q)}^* q) - dZ(q)(\iota^* \xi'(q) - ad_{Z'(q)}^* q) \\ &\quad - [Z, Z'](q) + \langle dR(q)(\cdot)\xi(q), \xi'(q) \rangle) \end{aligned} \quad (2.9)$$

where $\xi, \xi' : U \longrightarrow \mathfrak{g}^*$, $Z, Z' : U \longrightarrow \mathfrak{h}$ are smooth (resp. holomorphic) maps and $\langle dR(q)(\cdot)\xi(q), \xi'(q) \rangle$ is the element in \mathfrak{h} whose pairing with $\lambda \in \mathfrak{h}^*$ is $\langle dR(q)(\lambda)\xi(q), \xi'(q) \rangle$. We shall call $(A^*\Gamma, [\cdot, \cdot]_{A^*\Gamma}, a_*)$ the *coboundary dynamical Lie algebroid* associated to R .

Now, for any Lie algebroid $(A, [\cdot, \cdot]_A, a_A)$ over a smooth manifold M , recall that there exists a Lie-Poisson structure on the dual bundle A^* [CDW] which is uniquely determined by the property

$$\{l_X, l_Y\} = l_{[X, Y]_A} \quad (2.10)$$

where for $X, Y \in Sect(M, A)$, l_X and l_Y are the corresponding linear functions on A^* . The following result was obtained in [L2].

Theorem 2.2. (a) *The Lie-Poisson structure on the dual bundle $A\Gamma$ of the coboundary dynamical Lie algebroid $(A^*\Gamma, [\cdot, \cdot]_{A^*\Gamma}, a_*)$ is given by*

$$\begin{aligned} & \{\varphi, \psi\}_{A\Gamma}(q, \lambda, X) \\ &= - \langle \lambda, [\delta_2 \varphi, \delta_2 \psi] \rangle + \langle dR(q)(\lambda) \delta \varphi, \delta \psi \rangle \\ & \quad + \langle X, -ad_{R(q)\delta \varphi - \delta_2 \varphi}^* \delta \psi + ad_{R(q)\delta \psi - \delta_2 \psi}^* \delta \varphi \rangle \\ & \quad - \langle q, [\delta_2 \varphi, \delta_1 \psi] + [\delta_1 \varphi, \delta_2 \psi] \rangle + \langle \delta_1 \psi, \iota^* \delta \varphi \rangle - \langle \delta_1 \varphi, \iota^* \delta \psi \rangle. \end{aligned} \quad (2.11)$$

(b) *With the action of H on $A\Gamma$ defined by the formula*

$$h \cdot (q, \lambda, X) = (Ad_{h^{-1}}^* q, Ad_{h^{-1}}^* \lambda, Ad_h X), \quad (2.12)$$

the dual bundle $A\Gamma$ of the coboundary dynamical Lie algebroid A^Γ equipped with the Lie-Poisson structure is a Hamiltonian H -space with equivariant momentum map*

$$\gamma : A\Gamma \longrightarrow \mathfrak{h}^*, \quad (q, \lambda, X) \mapsto \lambda. \quad (2.13)$$

In the rest of the section, we shall assume that \mathfrak{g} is a Lie algebra with a nondegenerate invariant pairing (\cdot, \cdot) and $\mathfrak{h} \subset \mathfrak{g}$ is a non-degenerate (i.e. $(\cdot, \cdot)_{\mathfrak{h} \times \mathfrak{h}}$ is non-degenerate) abelian Lie subalgebra. Then we can make the identifications $\mathfrak{g}^* \simeq \mathfrak{g}$, $\mathfrak{h}^* \simeq \mathfrak{h}$, $ad^* \simeq -ad$, $\iota^* \simeq \Pi_{\mathfrak{h}}$, where $\Pi_{\mathfrak{h}}$ is the projection map to \mathfrak{h} relative to the direct sum decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}^\perp$. Hence we can regard $R(q)$ as taking values in $End(\mathfrak{g})$. In this case, an important sufficient condition for an H -equivariant map R to define a coboundary dynamical Poisson groupoid is given by the modified dynamical Yang-Baxter equation (mDYBE):

$$\begin{aligned} & [R(q)X, R(q)Y] - R(q)([R(q)X, Y] + [X, R(q)Y]) \\ & + dR(q)\Pi_{\mathfrak{h}}X(Y) - dR(q)\Pi_{\mathfrak{h}}Y(X) + (dR(q)(\cdot)X, Y) \\ & = -c^2[X, Y], \end{aligned} \quad (2.14)$$

where c is a nonzero constant.

We now turn our attention to a subclass of $(\Gamma, \{\cdot, \cdot\}_R)$ and $(A^*\Gamma, [\cdot, \cdot]_{A^*\Gamma}, a_*)$ which are associated with so-called classical dynamical r-matrices with spectral parameter. In the following, we shall assume that \mathfrak{g} is a Lie algebra over \mathbb{C} .

Definition 2.3 [EV],[LX]. A classical dynamical r-matrix with spectral parameter is a meromorphic map $r : \mathfrak{h} \times \mathbb{C} \longrightarrow \mathfrak{g} \otimes \mathfrak{g}$ with a simple pole at $z = 0$ satisfying the following conditions for all $(q, z) \in \mathfrak{h} \times \mathbb{C}$ away from the poles of r ,

1. the zero weight condition:

$$[h \otimes 1 + 1 \otimes h, r(q, z)] = 0, \quad (2.15)$$

for all $h \in \mathfrak{h}$,

2. the generalized unitarity condition:

$$r^{12}(q, z) + r^{21}(q, -z) = 0, \quad (2.16)$$

3. the residue condition:

$$\text{Res}_{z=0} r(q, z) = \Omega, \quad (2.17)$$

where $\Omega \in (S^2 \mathfrak{g})^{\mathfrak{g}}$ is the Casimir element corresponding to (\cdot, \cdot) ,

4. the classical dynamical Yang-Baxter equation (CDYBE) with spectral parameter:

$$\begin{aligned} \text{Alt}(dr) + [r^{12}(q, z_{12}), r^{13}(q, z_{13}) + r^{23}(q, z_{23})] \\ + [r^{13}(q, z_{13}), r^{23}(q, z_{23})] = 0, \end{aligned} \quad (2.19)$$

where $z_{ij} = z_i - z_j$.

Let $L\mathfrak{g}$ be the loop algebra consisting of Laurent series with coefficients in \mathfrak{g} . If r is a classical dynamical r -matrix with spectral parameter, we define

$$(R(q)X)(z) = p.v. \frac{1}{2\pi i} \oint_C (r(q, w - z), X(w) \otimes 1) dw, \quad X \in L\mathfrak{g}, \quad (2.20)$$

where C is a small circle centered at 0 with positive orientation, and $p.v.$ denotes the principal value of the improper integral. We have the following result [LX].

Theorem 2.4. (a) R is an H -equivariant classical dynamical r -matrix which satisfies the $mDYBE$ with $c = -\frac{1}{4}$.

(b) For $X \in L\mathfrak{g}$, we have the formula

$$(R(q)X)(z) = \frac{1}{2}X(z) + \sum_{k \geq 0} \frac{1}{k!} \left(\frac{\partial^k r}{\partial z^k}(q, -z), X_{-(k+1)} \otimes 1 \right).$$

We now fix an open connected set $U \subset \mathfrak{h}$ on which R is holomorphic. Let $A\Omega = U \times \mathfrak{h} \times \mathfrak{g}$ be the trivial Lie algebroid over U with vertex algebra \mathfrak{g} . We shall identify its dual bundle $A^*\Omega$ with $U \times \mathfrak{h} \times \mathfrak{g}$ and equip it with the Lie-Poisson structure. On the other hand, we can use R in Theorem 2.4 above to construct the associated coboundary dynamical Lie algebroid $A^*\Gamma \simeq U \times \mathfrak{h} \times L\mathfrak{g}$. Therefore, we can equip its dual bundle $A\Gamma$ with the corresponding Lie-Poisson structure. For each $q \in U$, we now define a map $r_-^\#(q) : \mathfrak{g} \longrightarrow L\mathfrak{g}$ by the formula

$$((r_-^\#(q)\xi)(z), \eta) = (r(q, z), \eta \otimes \xi) \quad (2.21)$$

where $\xi, \eta \in \mathfrak{g}$.

Theorem 2.5 [LX]. *The map $\rho : A^*\Omega \longrightarrow A\Gamma$ given by*

$$(q, p, \xi) \mapsto (q, -\Pi_b \xi, p + r_-^\#(q)\xi)$$

is an H -equivariant Poisson map, where H acts on A^Ω by $h \cdot (q, p, \xi) = (q, p, Ad_h \xi)$.*

Definition 2.6. Let r be a classical dynamical r-matrix with spectral parameter and let $L = Pr_3 \circ \rho$, where ρ is the realization map in Theorem 2.5. Then the (complex holomorphic) Hamiltonian system on $A^*\Omega$ generated by the Hamiltonian function

$$\mathcal{H}^C(q, p, \xi) = \frac{1}{2} \oint_C (L(q, p, \xi), L(q, p, \xi)) \frac{dz}{2\pi iz} \quad (2.22)$$

is called the spin Calogero-Moser system associated with r . Here, C is a small circle centered at 0 with the positive orientation.

Remark 2.7. Actually we will use the real version of Theorem 2.5 in our application in Section 4 below.

3. Dirac reduction of Poisson maps and geometric construction of integrable systems via successive reductions.

The goal of this section is to develop a general scheme of constructing integrable systems based on realization in symmetric coboundary dynamical Lie algebroids and symmetric coboundary dynamical Poisson groupoids. In order to do this, we have to consider the method of Dirac reduction. We begin by recalling the notion of a Dirac submanifold as recently introduced in [X]. It is a generalization of the notion of cosymplectic submanifolds of Weinstein [W1]. For convenience, we shall formulate our results in this section in the differentiable category, but it will be clear that the results are also valid for the holomorphic category.

Definition 3.1. Let (P, π) be a Poisson manifold. A submanifold Q of P is a Dirac submanifold iff there exists a Whitney sum decomposition

$$T_Q P = TQ \oplus V_Q \quad (3.1)$$

where V_Q^\perp is a Lie subalgebroid of the cotangent Lie algebroid T^*P .

If Q is a Dirac submanifold of (P, π) , then necessarily Q carries a natural Poisson structure π_Q whose symplectic leaves are given by the intersection of Q with the symplectic leaves of P . Indeed, $\pi_Q^\# : T^*Q \longrightarrow TQ$ is just the anchor map of the Lie

subalgebroid $T^*Q \simeq V_Q^\perp$ of T^*P . Moreover, from the knowledge of the injective Lie algebroid morphism $T^*Q \longrightarrow T^*P$, it is easy to show that

$$\pi_Q^\# = pr \circ \pi^\#|_Q \circ pr^* \quad (3.2a)$$

where $pr : T_Q P \longrightarrow TQ$ is the projection map induced by the decomposition in (3.1) and pr^* is its dual. Alternatively, we have

$$\pi|_Q = \pi_Q + \pi' \quad (3.2b)$$

where $\pi' \in \text{Sect}(\wedge^2 V_Q)$.

We shall call Q equipped with the induced Poisson structure a *Dirac reduction* of P .

Remark 3.2. An important class of Dirac submanifolds is given by the cosymplectic submanifolds of Weinstein [W1], in which case $V_Q = \pi^\#|_Q ((TQ)^\perp)$. Note that when P is symplectic, the cosymplectic submanifolds of P are precisely its symplectic submanifolds. We shall give another important class of examples in Proposition 3.5 below.

Since we will be dealing with realization maps into the dual bundles of symmetric coboundary dynamical Lie algebroids (resp. symmetric coboundary dynamical Poisson groupoids), the following result is fundamental in reducing such maps.

Theorem 3.3. *Let $\phi : P_1 \longrightarrow P_2$ be a Poisson map and let $Q_1 \subset P_1$, $Q_2 \subset P_2$ be Dirac submanifolds with respective Whitney sum decompositions*

$$T_{Q_1} P_1 = TQ_1 \oplus V_{Q_1}, \quad T_{Q_2} P_2 = TQ_2 \oplus V_{Q_2}.$$

Then under the assumptions that

(i) $\phi(Q_1) \subset Q_2$,

(ii) $T_x \phi(V_{Q_1})_x \subset (V_{Q_2})_{\phi(x)}$, $\forall x \in Q_1$,

the map $\phi|_{Q_1} : Q_1 \longrightarrow Q_2$ is a Poisson map, when Q_1 and Q_2 are equipped with the induced Poisson structures.

Proof. Let $\pi_{P_i}^\#, \pi_{Q_i}^\#$ be the bundle maps associated with the Poisson structures on P_i, Q_i , $i = 1, 2$. Since Q_1 is a Dirac submanifold of P_1 , we have

$$\pi_{P_1}^\#(x)(\alpha) = \pi_{Q_1}^\#(x)(\alpha|_{T_x Q_1}) + \tilde{\pi}_{Q_1}^\#(x)(\alpha|(V_{Q_1})_x) \quad (*)$$

for all $x \in Q_1$, $\alpha \in T_x^* P_1$, where $\tilde{\pi}_{Q_1}^\# : V_{Q_1}^* \longrightarrow V_{Q_1}$. Similarly,

$$\pi_{P_2}^\#(y)(\beta) = \pi_{Q_2}^\#(y)(\beta|_{T_y Q_2}) + \tilde{\pi}_{Q_2}^\#(y)(\beta|(V_{Q_2})_y) \quad (**)$$

for all $y \in Q_2$, $\beta \in T_y^* P_2$, where $\tilde{\pi}_{Q_2}^\sharp : V_{Q_2}^* \longrightarrow V_{Q_2}$. Now, it follows from (*) that

$$\begin{aligned} & T_x \phi \cdot \pi_{P_1}^\sharp(x) \cdot T_x^* \phi(\beta) \\ &= T_x \phi \cdot \pi_{Q_1}^\sharp(x)(T_x^* \phi(\beta) \mid T_x Q_1) + T_x \phi \cdot \tilde{\pi}_{Q_1}^\sharp(x)(T_x^* \phi(\beta) \mid (V_{Q_1})_x) \end{aligned}$$

for all $x \in Q_1$, $\beta \in T_{\phi(x)}^* P_2$. From assumption (i), we have $T_x \phi \cdot \pi_{Q_1}^\sharp(x)(T_x^* \phi(\beta) \mid T_x Q_1) \in T_{\phi(x)} Q_2$. On the other hand, assumption (ii) implies that $T_x \phi \cdot \tilde{\pi}_{Q_1}^\sharp(x)(T_x^* \phi(\beta) \mid (V_{Q_1})_x) \in (V_{Q_2})_{\phi(x)}$. Since ϕ is Poisson, it follows from (**) above that we also have

$$\begin{aligned} & T_x \phi \cdot \pi_{P_1}^\sharp(x) \cdot T_x^* \phi(\beta) \\ &= \pi_{Q_2}^\sharp(\phi(x))(\beta \mid T_{\phi(x)} Q_2) + \tilde{\pi}_{Q_2}^\sharp(\phi(x))(\beta \mid (V_{Q_2})_{\phi(x)}) \end{aligned}$$

for all $x \in Q_1$, $\beta \in T_{\phi(x)}^* P_2$. Therefore, upon equating the two expressions for $T_x \phi \cdot \pi_{P_1}^\sharp(x) \cdot T_x^* \phi(\beta)$, we obtain

$$\begin{aligned} & T_x \phi \cdot \pi_{Q_1}^\sharp(x)(T_x^* \phi(\beta) \mid T_x Q_1) \\ &= \pi_{Q_2}^\sharp(\phi(x))(\beta \mid T_{\phi(x)} Q_2) \end{aligned}$$

which shows that $\phi \mid Q_1 : Q_1 \longrightarrow Q_2$ is Poisson, as desired. \square

Definition 3.4. The map $\phi \mid Q_1 : Q_1 \longrightarrow Q_2$ in the theorem above will be called a *Dirac reduction* of the Poisson map $\phi : P_1 \longrightarrow P_2$.

The following result gives an important class of Dirac submanifolds which plays a key role in this work.

Proposition 3.5 [X]. *Let $\sigma : P \longrightarrow P$ be a Poisson involution, i.e., an involution which is also a Poisson map. Then its stable locus Q is a Dirac submanifold of P with $V_Q = \bigcup_{x \in Q} \ker(T_x \sigma + 1)$.*

As a consequence of this result, the stable locus of a Poisson involution carries a natural Poisson structure. This fact was also noted in [FP] and was implicit in the earlier work of several authors [Bon], [Boal]. (See also p.194 of [RSTS].)

In the special case when the Dirac submanifolds in Theorem 3.3 are the stable loci of Poisson involutions, we have the following result.

Corollary 3.6. *Let $\sigma_1 : P_1 \longrightarrow P_1$, $\sigma_2 : P_2 \longrightarrow P_2$ be Poisson involutions with stable loci given by Q_1 and Q_2 respectively. If $\phi : P_1 \longrightarrow P_2$ is a Poisson map which commutes with σ_1 , σ_2 , i.e. $\sigma_2 \circ \phi = \phi \circ \sigma_1$, then $\phi \mid Q_1 : Q_1 \longrightarrow Q_2$ is a Poisson map, when Q_1 and Q_2 are equipped with the induced structures.*

Proof. Under the assumption that ϕ commutes with the Poisson involutions, it is easy to check that the conditions in Theorem 3.3 are satisfied with

$$V_{Q_1} = \bigcup_{x \in Q_1} \ker(T_x \sigma_1 + 1), \quad V_{Q_2} = \bigcup_{y \in Q_2} \ker(T_y \sigma_2 + 1).$$

Hence the assertion follows. \square

We next consider the problem of reducing a Hamiltonian G -space P to a Dirac submanifold Q . Note that Q is in general not a G -space. So a natural question is: under what condition is Q a Hamiltonian H -space for some Lie subgroup $H \subset G$?

Proposition 3.7. *Let $\Phi : G \times P \longrightarrow P$ be a Hamiltonian group action of G on the Poisson manifold (P, π) , and let Q be a Dirac submanifold of P with Whitney sum decomposition $T_Q P = TQ \oplus V_Q$. If H is a Lie subgroup of G with $\text{Lie}(H) = \mathfrak{h}$ and if the action Φ induces an action of H on Q satisfying*

$$T_x \Phi_h(V_x) \subset V_{\Phi_h(x)}, \quad \forall h \in H, x \in Q,$$

where $V_Q = \bigcup_{q \in Q} V_q$, then the H -action on Q is also a Hamiltonian group action. Moreover, if $J : P \longrightarrow \mathfrak{g}^*$ is a G -equivariant momentum map for Φ , then the map $J_Q = i^* \circ (J|_Q) : Q \longrightarrow \mathfrak{h}^*$ is a H -equivariant momentum map for the H -action on Q . Here, i^* is the dual map of the Lie inclusion $i : \mathfrak{h} \longrightarrow \mathfrak{g}$.

Proof. For $h \in H$, the assertion that $\Phi_h|_Q : Q \longrightarrow Q$ is Poisson is a consequence of Theorem 3.3. To show that J_Q is an equivariant momentum map for the H action on Q , note that

$$\left. \frac{d}{dt} \right|_{t=0} \Phi_{e^{tZ}}(x) = \pi_Q^\#(x)(d\hat{J}(Z)(x) | T_x Q)$$

for all $Z \in \mathfrak{h}$ and $x \in Q$. But it is easy to check that the map $\hat{J}_Q : \mathfrak{h} \longrightarrow C^\infty(Q)$ defined by $\hat{J}_Q(Z)(x) = \langle J_Q(x), Z \rangle$, $Z \in \mathfrak{h}$, $x \in Q$ satisfies $d\hat{J}_Q(x) = d\hat{J}(Z)(x) | T_x Q$. This completes the proof. \square

Corollary 3.8. *Let $\Phi : G \times P \longrightarrow P$ be a Hamiltonian group action of G on the Poisson manifold (P, π) and suppose $\sigma : P \longrightarrow P$ is a Poisson involution. If H is a Lie subgroup of G such that*

$$\Phi_h \circ \sigma = \sigma \circ \Phi_h, \quad \forall h \in H,$$

then Φ induces a Hamiltonian group action of H on the stable locus P^σ . Moreover, if $J : P \longrightarrow \mathfrak{g}^*$ is a G -equivariant momentum map for Φ , then the map $\tilde{J} = i^* \circ (J|_{P^\sigma}) : P^\sigma \longrightarrow \mathfrak{h}^*$ is a H -equivariant momentum map for the H -action on P^σ .

Proof. It follows from the assumption $\Phi_h \circ \sigma = \sigma \circ \Phi_h$, $h \in H$ that Φ induces an action of H on $Q = P^\sigma$. Let $V_Q = \bigcup_{x \in Q} \ker(T_x \sigma + 1)$, we want to show that $T_x \Phi_h(V_x) \subset V_{\Phi_h(x)}$, $h \in H, x \in Q$. For this purpose, take any $v \in V_x$. Then we have

$$\begin{aligned} & (T_{\Phi_h(x)} \sigma + 1) T_x \Phi_h(v) \\ &= T_x (\Phi_h \circ \sigma)(v) + T_x \Phi_h(v) \quad (\text{since } \Phi_h \circ \sigma = \sigma \circ \Phi_h) \\ &= 0 \end{aligned}$$

where we have used the property $T_x \sigma(v) = -v$ in the last step. Hence the assertion follows from Proposition 3.7. \square

We next discuss Poisson involutions on $(A\Gamma, \{\cdot, \cdot\}_{A\Gamma})$, where $\{\cdot, \cdot\}_{A\Gamma}$ is the Lie-Poisson structure in (2.11). The following result was obtained in [X] using Lie bialgebroid theory [MX]. We will give an elementary proof based on the explicit formula in (2.11).

Proposition 3.9. *Let $s : \mathfrak{g} \longrightarrow \mathfrak{g}$ be an involutive Lie algebra anti-morphism which preserves \mathfrak{h} and assume that $s \circ R(q) \circ s^* = -R(s_{\mathfrak{h}}^* q)$ for all $q \in U$ where $s_{\mathfrak{h}} = s|_{\mathfrak{h}}$. Then the map*

$$\sigma : (A\Gamma, \{\cdot, \cdot\}_{A\Gamma}) \longrightarrow (A\Gamma, \{\cdot, \cdot\}_{A\Gamma}), (q, \lambda, X) \mapsto (s_{\mathfrak{h}}^*(q), -s_{\mathfrak{h}}^*(\lambda), s(X)) \quad (3.3)$$

is a Poisson involution.

Proof. From the property that $s \circ R(q) \circ s^* = -R(s_{\mathfrak{h}}^* q)$, it follows that $s(dR(q)(\lambda)s^*\xi) = -dR(s_{\mathfrak{h}}^* q)(s_{\mathfrak{h}}^* \lambda)(\xi)$. The rest of the proof is plain. \square

Remark 3.10. The virtue of our direct verification in the above proof lies in the fact that it extends to more general constructions in which the Lie bialgebroid structure is lost.

We shall call $(A^*\Gamma, [\cdot, \cdot]_{A^*\Gamma}, a_*, \sigma^*)$ a *symmetric coboundary dynamical Lie algebroid*. In order to compute the Poisson structure on the stable locus, we shall introduce some notation which we shall use in the rest of the section. Let $(P, \{\cdot, \cdot\}_P)$ be a Poisson manifold and suppose $\tau : P \longrightarrow P$ is a Poisson involution with stable locus P^τ . Then for $\varphi \in C^\infty(P)$, we put $\tilde{\varphi} = \varphi|_{P^\tau}$ and $\varphi^\tau = \frac{1}{2}(\varphi + \tau^* \varphi)$. Since for $Q = P^\tau$, we have $V_Q = \bigcup_{x \in Q} \ker(T_x \tau + 1)$ in the Whitney sum decomposition for $T_Q P$. Hence it follows from (3.2) that the induced Poisson structure on P^τ is given by the formula

$$\{\tilde{\varphi}, \tilde{\psi}\}_{P^\tau}(x) = \{\varphi^\tau, \psi^\tau\}_P(x) \quad (3.4)$$

for $x \in P^\tau$ and $\varphi, \psi \in C^\infty(P)$.

Proposition 3.11. *The Poisson structure on the stable locus $A\Gamma^\sigma$ of the Poisson involution in (3.3) is given by*

$$\begin{aligned}
& \{\tilde{F}_1, \tilde{F}_2\}_{A\Gamma^\sigma}(q, \lambda, X) \\
&= -\langle \lambda, [\delta_2 \tilde{F}_1, \delta_2 \tilde{F}_2] \rangle + \langle dR(q)(\lambda) \delta \tilde{F}_1, \delta \tilde{F}_2 \rangle \\
&\quad + \langle X, -ad_{R(q)\delta \tilde{F}_1 - \delta_2 \tilde{F}_1}^* \delta \tilde{F}_2 + ad_{R(q)\delta \tilde{F}_2 - \delta_2 \tilde{F}_2}^* \delta \tilde{F}_1 \rangle \\
&\quad - \langle q, [\delta_2 \tilde{F}_1, \delta_1 \tilde{F}_2] + [\delta_1 \tilde{F}_1, \delta_2 \tilde{F}_2] \rangle + \langle \delta_1 \tilde{F}_2, \iota^* \delta \tilde{F}_1 \rangle - \langle \delta_1 \tilde{F}_1, \iota^* \delta \tilde{F}_2 \rangle
\end{aligned} \tag{3.5}$$

for $F_1, F_2 \in C^\infty(A\Gamma)$, $(q, \lambda, X) \in A\Gamma^\sigma$ where

$$\begin{aligned}
\delta_1 \tilde{F}_i &:= \frac{1}{2}(\delta_1 F_i + s_{\mathfrak{h}}(\delta_1 F_i)), & \delta_2 \tilde{F}_i &:= \frac{1}{2}(\delta_2 F_i - s_{\mathfrak{h}}(\delta_2 F_i)), \\
\delta \tilde{F}_i &:= \frac{1}{2}(\delta F_i + s^*(\delta F_i)), & i &= 1, 2.
\end{aligned} \tag{3.6}$$

Moreover, the Hamiltonian vector field on $A\Gamma^\sigma$ generated by \tilde{F} is of the form

$$\begin{aligned}
& X_{\tilde{F}}(q, \lambda, X) \\
&= (\iota^* \delta \tilde{F} - ad_{\delta_2 \tilde{F}}^* q, -ad_{\delta_2 \tilde{F}}^* \lambda + \iota^* ad_X^* \delta \tilde{F} - ad_{\delta_1 \tilde{F}}^* q, \\
&\quad [X, R(q)\delta \tilde{F} - \delta_2 \tilde{F}] + dR(q)(\lambda) \delta \tilde{F} - \delta_1 \tilde{F} + R(q)ad_X^* \delta \tilde{F}).
\end{aligned} \tag{3.7}$$

Proof. The bundle map π^\sharp corresponding to the Poisson bracket $\{\cdot, \cdot\}_{A\Gamma}$ in (2.11) is given by

$$\begin{aligned}
& \pi^\sharp(q, \lambda, X)(Z_1, Z_2, \xi) \\
&= (\iota^* \xi - ad_{Z_2}^* q, -ad_{Z_2}^* \lambda + \iota^* ad_X^* \xi - ad_{Z_1}^* q, \\
&\quad [X, R(q)\xi - Z_2] + dR(q)(\lambda)\xi - Z_1 + R(q)(ad_X^* \xi))
\end{aligned}$$

where $(q, \lambda, X) \in A\Gamma$ and $(Z_1, Z_2, \xi) \in \mathfrak{h} \times \mathfrak{h} \times \mathfrak{g}^* \simeq T_{(q, \lambda, X)}^*(A\Gamma)$. Let $Q = A\Gamma^\sigma$ and let $\iota_Q : Q \longrightarrow A\Gamma$ be the canonical inclusion. In this case, the bundle V_Q in the vector bundle decomposition $T(A\Gamma) = TQ \oplus V_Q$ is just the bundle of -1 eigenspaces of $T\sigma$. Therefore, if $(q, \lambda, X) \in Q$, and $(Z_1, Z_2, \xi) \in \mathfrak{h} \times \mathfrak{h} \times \mathfrak{g}^* \simeq T_{(q, \lambda, X)}^*(A\Gamma)$, it follows that the bundle map of the induced Poisson structure on Q is given by

$$\begin{aligned}
& \pi_Q^\sharp(q, \lambda, X)(T_{(q, \lambda, X)}^* \iota_Q(Z_1, Z_2, \xi)) \\
&= (\iota^* \tilde{\xi} - ad_{\tilde{Z}_2}^* q, -ad_{\tilde{Z}_2}^* \lambda + \iota^* ad_X^* \tilde{\xi} - ad_{\tilde{Z}_1}^* q, \\
&\quad [X, R(q)\tilde{\xi} - \tilde{Z}_2] + dR(q)(\lambda)\tilde{\xi} - \tilde{Z}_1 + R(q)(ad_X^* \tilde{\xi}))
\end{aligned}$$

where

$$\begin{aligned}\tilde{Z}_1 &= \frac{1}{2}(Z_1 + s_{\mathfrak{h}}(Z_1)), & \tilde{Z}_2 &= \frac{1}{2}(Z_2 - s_{\mathfrak{h}}(Z_2)), \\ \tilde{\xi} &= \frac{1}{2}(\xi + s^*(\xi)).\end{aligned}$$

Since $d\tilde{F}(q, \lambda, X) = T_{(q, \lambda, X)}^* \iota_Q dF(q, \lambda, X)$, the formula for the vector field $X_{\tilde{F}}$ is immediate from the above expression. On the other hand, the formula for the Poisson bracket is a consequence of (2.3) and (3.4) as we have $dF_i^\sigma(q, \lambda, X) = (\delta_1 \tilde{F}_i, \delta_2 \tilde{F}_i, \delta \tilde{F}_i)$, $i = 1, 2$. \square

We now turn to corresponding results for the coboundary dynamical Poisson groupoids. The following result was also obtained in [X] by invoking Lie bialgebroid theory. We can, of course, verify the assertion in a direct way by using the formula in (2.7).

Proposition 3.12. *Let R be an H -equivariant classical dynamical r -matrix such that $s \circ R(q) \circ s^* = -R(s_{\mathfrak{h}}^* q)$ for all $q \in U$ where s is as in Proposition 3.9. If $(\Gamma, \{\cdot, \cdot\}_R)$ is the coboundary dynamical Poisson groupoid associated to R , then the map*

$$\Sigma : (\Gamma, \{\cdot, \cdot\}_R) \longrightarrow (\Gamma, \{\cdot, \cdot\}_R), (u, g, v) \mapsto (s_{\mathfrak{h}}^*(v), S(g), s_{\mathfrak{h}}^*(u)) \quad (3.8)$$

is a Poisson involution, where $S : G \longrightarrow G$ is the group anti-morphism which integrates s .

We shall call $(\Gamma, \{\cdot, \cdot\}_R, \Sigma)$ a *symmetric coboundary dynamical Poisson groupoid*.

Proposition 3.13. *With the involution Σ in (3.8), the induced Poisson structure on its stable locus Γ^Σ is given by*

$$\begin{aligned}& \{\tilde{\varphi}, \tilde{\psi}\}_{\Gamma^\Sigma}(u, g, s_{\mathfrak{h}}^*(u)) \\ &= 2 \langle u, [\delta_1 \tilde{\varphi}, \delta_1 \tilde{\psi}] \rangle - 2 \langle \iota \delta_1 \tilde{\varphi}, D\tilde{\psi} \rangle + 2 \langle \iota \delta_1 \tilde{\psi}, D\tilde{\varphi} \rangle \\ & \quad + \langle R(s_{\mathfrak{h}}^*(u)) D' \tilde{\varphi}, D' \tilde{\psi} \rangle - \langle R(u) D\tilde{\varphi}, D\tilde{\psi} \rangle \\ &= 2 \langle u, [\delta_1 \tilde{\varphi}, \delta_1 \tilde{\psi}] \rangle - 2 \langle \iota \delta_1 \tilde{\varphi}, D\tilde{\psi} \rangle + 2 \langle \iota \delta_1 \tilde{\psi}, D\tilde{\varphi} \rangle \\ & \quad - 2 \langle R(u) D\tilde{\varphi}, D\tilde{\psi} \rangle\end{aligned} \quad (3.9)$$

for $\varphi, \psi \in C^\infty(\Gamma)$, $(u, g, s_{\mathfrak{h}}^*(u)) \in \Gamma^\Sigma$, where

$$\begin{aligned}\delta_1 \tilde{\varphi} &:= \frac{1}{2}(\delta_1 \varphi + s_{\mathfrak{h}}(\delta_2 \varphi)), & D\tilde{\varphi} &:= \frac{1}{2}(D\varphi + s^*(D'\varphi)), \\ D'\tilde{\varphi} &:= \frac{1}{2}(D'\varphi + s^*(D\varphi)),\end{aligned} \quad (3.10)$$

and similarly for $\tilde{\psi}$. Here, $\delta_1\varphi, \delta_2\varphi$ are the partial derivatives of φ with respect to the variables in U and $D'\varphi, D\varphi$ are the left and right gradients of φ with respect to the variable in G . Hence the Hamiltonian vector field on Γ^Σ generated by $\tilde{\varphi}$ is of the form

$$\begin{aligned} & X_{\tilde{\varphi}}(u, g, s_{\mathfrak{h}}^*(u)) \\ &= (ad_{\delta_1\tilde{\varphi}}^*u + \iota^*D\tilde{\varphi}, -T_e r_g \delta_1\tilde{\varphi} - T_e l_g s_{\mathfrak{h}}^*(\delta_1\tilde{\varphi}) + T_e l_g R(s_{\mathfrak{h}}^*(u))s^*(D\tilde{\varphi}) \\ & \quad - T_e r_g R(u)D\tilde{\varphi}, s_{\mathfrak{h}}^*(ad_{\delta_1\tilde{\varphi}}^*u + \iota^*D\tilde{\varphi})). \end{aligned} \quad (3.11)$$

Proof. According to (2.7) and (3.4), we can express the bracket $\{\tilde{\varphi}, \tilde{\psi}\}_{\Gamma^\Sigma}$ in the following form:

$$\begin{aligned} \{\tilde{\varphi}, \tilde{\psi}\}_{\Gamma^\Sigma}(u, g, s_{\mathfrak{h}}^*(u)) &= \langle u, [\delta_1\tilde{\varphi}, \delta_1\tilde{\psi}] \rangle - \langle s_{\mathfrak{h}}^*(u), [\delta_2\tilde{\varphi}, \delta_2\tilde{\psi}] \rangle \\ & \quad - \langle \iota\delta_1\tilde{\varphi}, D\tilde{\psi} \rangle - \langle \iota\delta_2\tilde{\varphi}, D'\tilde{\psi} \rangle \\ & \quad + \langle \iota\delta_1\tilde{\psi}, D\tilde{\varphi} \rangle + \langle \iota\delta_2\tilde{\psi}, D'\tilde{\varphi} \rangle \\ & \quad + \langle R(s_{\mathfrak{h}}^*(u))D'\tilde{\varphi}, D'\tilde{\psi} \rangle - \langle R(u)D\tilde{\varphi}, D\tilde{\psi} \rangle \end{aligned}$$

where $\delta_2\tilde{\varphi} := \frac{1}{2}(\delta_2\varphi + s_{\mathfrak{h}}(\delta_1\varphi))$ and the other derivatives are defined in (3.10). However, it is easy to show that

$$\langle s_{\mathfrak{h}}^*(u), [\delta_2\tilde{\varphi}, \delta_2\tilde{\psi}] \rangle = - \langle u, [\delta_1\tilde{\varphi}, \delta_1\tilde{\psi}] \rangle, \quad \langle \iota\delta_2\tilde{\varphi}, D'\tilde{\psi} \rangle = \langle \iota\delta_1\tilde{\varphi}, D\tilde{\psi} \rangle$$

and $\langle R(s_{\mathfrak{h}}^*(u))D'\tilde{\varphi}, D'\tilde{\psi} \rangle = - \langle R(u)D\tilde{\varphi}, D\tilde{\psi} \rangle$. Therefore, the above expression for the bracket simplifies to the ones in the statement of the theorem. The computation of the vector field proceeds as in the proof of Proposition 3.11 and so we skip the details. \square

Now let us recall from [L2] that the Lie-Poisson structure on the dual bundle $A^*\Gamma$ of the trivial Lie algebroid $A\Gamma$ is given by $\{\mathcal{F}, \mathcal{G}\}_{A^*\Gamma}(q, p, \xi) = \langle \delta_2\mathcal{F}, \delta_1\mathcal{G} \rangle - \langle \delta_1\mathcal{F}, \delta_2\mathcal{G} \rangle + \langle \xi, [\delta\mathcal{F}, \delta\mathcal{G}] \rangle$.

We shall leave the proof of the next result to the reader.

Proposition 3.14. *Let $b : \mathfrak{h} \longrightarrow \mathfrak{h}$ be an involutive linear map and suppose $c : \mathfrak{g} \longrightarrow \mathfrak{g}$ is an involutive Lie algebra morphism. Then the map*

$$\theta : (A^*\Gamma, \{\cdot, \cdot\}_{A^*\Gamma}) \longrightarrow (A^*\Gamma, \{\cdot, \cdot\}_{A^*\Gamma}), (q, p, \xi) \mapsto (b^*(q), b(p), c^*(\xi)) \quad (3.12)$$

is a Poisson involution. Moreover, the induced Poisson structure on the stable locus A^Γ^θ is given by*

$$\begin{aligned} & \{\tilde{\mathcal{F}}, \tilde{\mathcal{G}}\}_{A^*\Gamma^\theta}(q, p, \xi) \\ &= \langle \delta_2\tilde{\mathcal{F}}, \delta_1\tilde{\mathcal{G}} \rangle - \langle \delta_1\tilde{\mathcal{F}}, \delta_2\tilde{\mathcal{G}} \rangle + \langle \xi, [\delta\tilde{\mathcal{F}}, \delta\tilde{\mathcal{G}}] \rangle \end{aligned} \quad (3.13)$$

for $\mathcal{F}, \mathcal{G} \in C^\infty(A^*\Gamma)$ and $(q, p, \xi) \in A^*\Gamma^\theta$ where

$$\begin{aligned}\delta_1 \tilde{\mathcal{F}} &:= \frac{1}{2}(\delta_1 \mathcal{F} + b(\delta_1 \mathcal{F})), & \delta_2 \tilde{\mathcal{F}} &:= \frac{1}{2}(\delta_2 \mathcal{F} + b^*(\delta_2 \mathcal{F})), \\ \delta \tilde{\mathcal{F}} &:= \frac{1}{2}(\delta \mathcal{F} + c(\delta \mathcal{F})),\end{aligned}\tag{3.14}$$

and similarly for $\tilde{\mathcal{G}}$.

Thus the induced structure $\{\cdot, \cdot\}_{A^*\Gamma^\theta}$ is still a product structure. Indeed, under the natural isomorphism between $(\mathfrak{g}^*)^{c^*}$ and $(\mathfrak{g}^c)^*$, we can identify the bracket on the stable locus $(\mathfrak{g}^*)^{c^*}$ with the Lie-Poisson structure on $(\mathfrak{g}^c)^*$.

We are now ready to formulate the main results of this section. In what follows, let X be a Hamiltonian H -space (the H -action will be denoted by \mathcal{C}) with equivariant momentum map $J : X \longrightarrow \mathfrak{g}^*$, and let $\kappa : X \longrightarrow X$ be a Poisson involution on X . Beginning with H -invariant Hamiltonian systems on X which admit either a realization in $(A\Gamma, \{\cdot, \cdot\}_{A\Gamma})$ or $(\Gamma, \{\cdot, \cdot\}_R)$, we shall show how reduction to Dirac submanifolds followed by Poisson reduction can lead us to integrable systems.

Case 1. The case of realization in $(A\Gamma, \{\cdot, \cdot\}_{A\Gamma})$

Let $\rho : X \longrightarrow A\Gamma$ be a realization of the Poisson manifold X in the dual bundle $A\Gamma$ of the Lie algebroid $A^*\Gamma$; i.e., ρ is a Poisson map. Let us recall from Theorem 2.2 that $A\Gamma$ with the action

$$\mathcal{A} : H \times A\Gamma \longrightarrow A\Gamma, \quad \mathcal{A}_h(q, \lambda, \xi) = (Ad_{h^{-1}}^* q, Ad_{h^{-1}}^* \lambda, Ad_h \xi) \tag{3.15}$$

is a Hamiltonian H -space with equivariant momentum map

$$\gamma : A\Gamma \longrightarrow \mathfrak{h}^*, \quad (q, \lambda, \xi) \mapsto \lambda. \tag{3.16}$$

We begin by making the following assumption:

A1. there exists a Poisson involution

$$\sigma : (A\Gamma, \{\cdot, \cdot\}_{A\Gamma}) \longrightarrow (A\Gamma, \{\cdot, \cdot\}_{A\Gamma}), \quad (q, \lambda, \xi) \mapsto (s_{\mathfrak{h}}^*(q), -s_{\mathfrak{h}}^*(\lambda), s(\xi)) \tag{3.17}$$

on $A\Gamma$ (where s satisfies the assumptions in Proposition 3.9) such that

$$\sigma \circ \rho = \rho \circ \kappa. \tag{3.18}$$

Then according to Corollary 3.6, the map ρ restricts to a Poisson map $\tilde{\rho} : X^\kappa \longrightarrow A\Gamma^\sigma$, when X^κ and $A\Gamma^\sigma$ are equipped with the induced structures. Thus the stable locus X^κ admits a realization in $A\Gamma^\sigma \simeq U_s \times \mathfrak{h}_s^* \times \mathfrak{g}^s$, where

$$U_s = \{q \in U \mid s_{\mathfrak{h}}^*(q) = q\}, \tag{3.19}$$

$$\mathfrak{h}_s^* = \{\lambda \in \mathfrak{h}^* \mid s^*(\lambda) = -\lambda\}, \quad (3.20)$$

and \mathfrak{g}^s is the fixed point set of s . Let $I(\mathfrak{g})$ be the ring of ad-invariant functions on \mathfrak{g} , and let $I(\mathfrak{g}^s)$ consists of the restrictions of functions in $I(\mathfrak{g})$ to \mathfrak{g}^s . If Pr_3 denote the projection map from $A\Gamma^\sigma \simeq U_s \times \mathfrak{h}_s^* \times \mathfrak{g}^s$ to the factor \mathfrak{g}^s , then a natural family of invariant functions on $A\Gamma^\sigma$ is $Pr_3^* I(\mathfrak{g}^s)$. Our first result on Poisson commuting functions will have application in Section 4.2 below.

Theorem 3.15. *Let σ be a Poisson involution of the form in (3.17) on $A\Gamma$ (where s satisfies the assumptions in Proposition 3.9) and suppose $\mathfrak{h}_s^* = \{0\}$, then the functions in $Pr_3^* I(\mathfrak{g}^s)$ Poisson commute in $A\Gamma^\sigma$. Consequently, if we assume in addition that A1 is valid, then $\tilde{\rho}^* Pr_3^* I(\mathfrak{g}^s)$ is a Poisson commuting family of functions on X^κ , where $\tilde{\rho} = \rho \mid X^\kappa$.*

Proof. Let $f_1, f_2 \in I(\mathfrak{g})$, and let \tilde{f}_1, \tilde{f}_2 be their restrictions to $I(\mathfrak{g}^s)$. Then from (3.5), we have

$$\begin{aligned} & \{Pr_3^* \tilde{f}_1, Pr_3^* \tilde{f}_2\}_{A\Gamma^\sigma}(q, 0, \xi) \\ &= \widetilde{\{Pr_3^* f_1, Pr_3^* f_2\}}_{A\Gamma^\sigma}(q, 0, \xi) \\ &= \langle \xi, -ad_{R(q)\delta\widetilde{Pr_3^* f_1}}^* \delta\widetilde{Pr_3^* f_2} + ad_{R(q)\delta\widetilde{Pr_3^* f_2}}^* \delta\widetilde{Pr_3^* f_1} \rangle \end{aligned}$$

where in the last two lines, we have used the same symbol Pr_3 to denote the projection map from $A\Gamma$ to \mathfrak{g} and $\widetilde{Pr_3^* f_i}$ denote the restriction of $Pr_3^* f_i$ to $A\Gamma^\sigma$, $i = 1, 2$. Now, by direct calculation, we find

$$\delta\widetilde{Pr_3^* f_i} = \frac{1}{2}(df_i + s^*(df_i)), \quad i = 1, 2.$$

Therefore, upon substituting into the above expression, we obtain

$$\begin{aligned} & \{Pr_3^* \tilde{f}_1, Pr_3^* \tilde{f}_2\}_{A\Gamma^\sigma}(q, 0, \xi) \\ &= \frac{1}{4} \langle [\xi, R(q)(df_1 + s^*(df_1))], df_2 + s^*(df_2) \rangle - (1 \leftrightarrow 2) \\ &= \frac{1}{4} \langle R(q)(df_1 + s^*(df_1)), ad_\xi^* df_2 + ad_\xi^* s^*(df_2) \rangle - (1 \leftrightarrow 2). \end{aligned}$$

But as $\xi \in \mathfrak{g}^s$, we have $ad_\xi^* \circ s^* = -s^* \circ ad_\xi^*$. Hence the first assertion follows from the fact that $ad_\xi^* df_i = 0$, $i = 1, 2$. The second assertion is now clear as assumption A1 implies that $\tilde{\rho}$ is Poisson by Corollary 3.6. \square

In the general case when $\mathfrak{h}_s^* \neq \{0\}$, the functions in $Pr_3^* I(\mathfrak{g}^s)$ is no longer a Poisson commuting family on $A\Gamma^\sigma$. Indeed, by a computation similar to the one in

the proof of the above theorem, we have

$$\begin{aligned} & \{Pr_3^* \tilde{f}_1, Pr_3^* \tilde{f}_2\}_{A\Gamma^\sigma}(q, \lambda, \xi) \\ &= \frac{1}{4} < dR(q)(\lambda)(df_1 + s^*(df_1)), df_2 + s^*(df_2) > \end{aligned} \quad (3.21)$$

for $f_1, f_2 \in I(\mathfrak{g})$. Nevertheless, it is clear from this expression that when we restrict to the submanifold $U_s \times \{0\} \times \mathfrak{g}^s$ of $A\Gamma^\sigma$, the bracket vanishes. Note that in general neither X^κ nor $A\Gamma^\sigma$ are Hamiltonian H -spaces. We now discuss a situation where we can obtain Poisson commuting functions on a reduced phase space. Motivated by our application in Section 4.1 below, we shall make the following assumptions to prepare the way for Poisson reduction:

- A2. the realization map ρ is H -equivariant,
- A3. for some Lie subgroup D of H ,

$$\mathcal{A}_d \circ \sigma = \sigma \circ \mathcal{A}_d, \quad \mathcal{C}_d \circ \kappa = \kappa \circ \mathcal{C}_d, \quad \forall d \in D, \quad (3.22)$$

- A4. if $\mathfrak{d} = \text{Lie}(D)$ and \mathfrak{h}_s^* is as in (3.20), we assume

$$\mathfrak{h}_s^* \subset \mathfrak{d}^*. \quad (3.23)$$

Proposition 3.16. *Under assumptions A1-A4, the stable loci X^κ , $A\Gamma^\sigma$ are Hamiltonian D -spaces with equivariant momentum maps $\tilde{J} = i_{\mathfrak{d}}^* \circ (J \mid X^\kappa)$ and $\tilde{\gamma} = \gamma \mid A\Gamma^\sigma$ respectively, where $i_{\mathfrak{d}} : \mathfrak{d} \longrightarrow \mathfrak{h}$ in the natural inclusion and $i_{\mathfrak{d}}^*$ is its dual map. Moreover, the map*

$$\tilde{\rho} = \rho \mid X^\kappa : X^\kappa \longrightarrow A\Gamma^\sigma \quad (3.24)$$

is a D -equivariant Poisson map.

Proof. From A3 and Corollary 3.8, it follows that the actions \mathcal{A} and \mathcal{C} induce Hamiltonian group actions of D on the stable loci $A\Gamma^\sigma$ and X^κ respectively. Using the second part of the same corollary and A4, we can easily obtain the equivariant momentum maps of these induced actions. Finally, that the map $\tilde{\rho}$ is well-defined and Poisson is a consequence of A1 and Corollary 3.6, and its D -equivariance is obvious from A2. \square

The above proposition completes the first stage of our reduction process in the general case and prepares the way for Poisson reduction. In order to obtain Poisson commuting functions in this general case, we shall make an additional assumption:

A5. for some regular value $\mu \in \mathfrak{d}^*$ of \tilde{J} ,

$$\tilde{\rho}(\tilde{J}^{-1}(\mu)) \subset \tilde{\gamma}^{-1}(0) = U_s \times \{0\} \times \mathfrak{g}^s. \quad (3.25)$$

With this additional assumption, we will construct the integrable systems and their realizations by Poisson reduction of the map $\tilde{\rho}$ in Proposition 3.16. Let D_μ be the isotropy subgroup of D for the D -action on X^κ , then by Poisson reduction [MR],[OR], the variety $X_\mu^\kappa = \tilde{J}^{-1}(\mu)/D_\mu$ inherits a unique Poisson structure $\{\cdot, \cdot\}_{X_\mu^\kappa}$ satisfying

$$\pi_\mu^*\{f_1, f_2\}_{X_\mu^\kappa} = i_\mu^*\{f'_1, f'_2\}_{X^\kappa}. \quad (3.26)$$

Here, $i_\mu : \tilde{J}^{-1}(\mu) \rightarrow X^\kappa$ is the inclusion map, $\pi_\mu : \tilde{J}^{-1}(\mu) \rightarrow X_\mu^\kappa$ is the canonical projection, $f_1, f_2 \in C^\infty(X_\mu^\kappa)$, and f'_1, f'_2 are (locally defined) smooth extensions of $\pi_\mu^*f_1, \pi_\mu^*f_2$ with differentials vanishing on the tangent spaces of the D -orbits. Similarly, we have the Poisson variety

$$(A\Gamma_0^\sigma = \tilde{\gamma}^{-1}(0)/D, \{\cdot, \cdot\}_{A\Gamma_0^\sigma}), \quad (3.27)$$

with the inclusion map $i_D : \tilde{\gamma}^{-1}(0) \rightarrow A\Gamma^\sigma$ and the canonical projection $\pi_D : \tilde{\gamma}^{-1}(0) \rightarrow A\Gamma_0^\sigma$. If Pr_i denotes the projection map onto the i -th factor of $U_s \times \mathfrak{h}_s^* \times \mathfrak{g}^s \simeq A\Gamma$, $i = 1, 2, 3$, we put

$$m = Pr_1 \circ \tilde{\rho} : X^\kappa \rightarrow U_s, \quad (3.28)$$

$$\tau = Pr_2 \circ \tilde{\rho} : X^\kappa \rightarrow \mathfrak{h}_s^*, \quad (3.29)$$

$$L = Pr_3 \circ \tilde{\rho} : X^\kappa \rightarrow \mathfrak{g}^s. \quad (3.30)$$

Clearly, functions in $i_D^* Pr_3^* I(\mathfrak{g}^s) \subset C^\infty(\tilde{\gamma}^{-1}(0))$ are D -invariant, hence they descend to functions in $C^\infty(A\Gamma_0^\sigma)$. On the other hand, it follows from Proposition 3.16 that the functions in $i_\mu^* L^* I(\mathfrak{g}^s) \subset C^\infty(\tilde{J}^{-1}(\mu))$ drop down to functions in $C^\infty(X_\mu^\kappa)$. Now, by Proposition 3.16 and assumption A5, it follows from Theorem 2.14 of [OR] that $\tilde{\rho}$ induces a unique Poisson map (called the reduction of $\tilde{\rho}$)

$$\hat{\rho} : X_\mu^\kappa \rightarrow A\Gamma_0^\sigma = (U_s \times \{0\} \times \mathfrak{g}^s)/D \quad (3.31)$$

characterized by $\pi_D \circ \tilde{\rho} \circ i_\mu = \hat{\rho} \circ \pi_\mu$. Hence X_μ^κ admits a realization in the Poisson variety $A\Gamma_0^\sigma$.

We shall use the following notation. For $f \in I(\mathfrak{g})$, the unique function in $C^\infty(A\Gamma_0^\sigma)$ determined by $i_D^* Pr_3^* \tilde{f}$ ($\tilde{f} = f|_{\mathfrak{g}^s}$) will be denoted by \bar{f} ; while the unique function in $C^\infty(X_\mu^\kappa)$ determined by $i_\mu^* L^* \tilde{f}$ will be denoted by \mathcal{F}_μ . From the definitions, we have

$$\mathcal{F}_\mu \circ \pi_\mu = (\hat{\rho}^* \bar{f}) \circ \pi_\mu = i_\mu^* L^* \tilde{f}. \quad (3.32)$$

Theorem 3.17. *If $\mathfrak{h}_s^* \neq \{0\}$, then under assumptions A1-A5, the map $\tilde{\rho} = \rho \mid X^\kappa : X^\kappa \longrightarrow A\Gamma^\sigma$ induces a unique Poisson map $\widehat{\rho} : X_\mu^\kappa \longrightarrow A\Gamma_0^\sigma$ such that*

- (a) *functions $\mathcal{F}_\mu = \widehat{\rho}^* \bar{f}$, $f \in I(\mathfrak{g})$, Poisson commute in $(X_\mu^\kappa, \{\cdot, \cdot\}_{X_\mu^\kappa})$,*
 (b) *if ψ_t is the induced flow on $\tilde{\gamma}^{-1}(0) = U_s \times \{0\} \times \mathfrak{g}^s$ generated by the Hamiltonian $Pr_3^* \bar{f}$, $f \in I(\mathfrak{g})$, and ϕ_t is the Hamiltonian flow of $\mathcal{F} = L^* \bar{f}$ on X^κ , then under the flow ϕ_t , we have*

$$\begin{aligned} \frac{d}{dt} m(\phi_t) &= \frac{1}{2} \iota^* (df(L(\phi_t)) + s^*(df(L(\phi_t))))), \\ \frac{d}{dt} \tau(\phi_t) &= 0, \\ \frac{d}{dt} L(\phi_t) &= \frac{1}{2} [L(\phi_t), R(m(\phi_t))(df(L(\phi_t)) + s^*(df(L(\phi_t))))] \\ &\quad + dR(m(\phi_t))(\tau(\phi_t))(df(L(\phi_t)) + s^*(df(L(\phi_t)))) \end{aligned}$$

where the term involving dR drops out on $\tilde{J}^{-1}(\mu)$. Moreover, the reduction ϕ_t^{red} of $\phi_t \circ i_\mu$ on X_μ^κ defined by $\phi_t^{red} \circ \pi_\mu = \pi_\mu \circ \phi_t \circ i_\mu$ is a Hamiltonian flow of $\mathcal{F}_\mu = \widehat{\rho}^* \bar{f}$ and $\widehat{\rho} \circ \phi_t^{red}(\pi_\mu(x)) = \pi_D \circ \psi_t(\tilde{\rho}(x))$, $x \in \tilde{J}^{-1}(\mu)$.

Proof. (a) Let $f_1, f_2 \in I(\mathfrak{g})$, then it is easy to check that $Pr_3^* \tilde{f}_1, Pr_3^* \tilde{f}_2$ are extensions of $\pi_D^* \bar{f}_1, \pi_D^* \bar{f}_2$ with differentials vanishing on the tangent spaces of the D -orbits. Therefore, if $x \in \tilde{J}^{-1}(\mu)$, we have $\tau(x) = 0$ by assumption A5 and hence

$$\begin{aligned} &\{\widehat{\rho}^* \bar{f}_1, \widehat{\rho}^* \bar{f}_2\}_{X_\mu^\kappa} \circ \pi_\mu(x) \\ &= \{\bar{f}_1, \bar{f}_2\}_{A\Gamma_0^\sigma} \circ \pi_D(\tilde{\rho}(x)) \\ &= \{Pr_3^* \tilde{f}_1, Pr_3^* \tilde{f}_2\}_{A\Gamma^\sigma}(\tilde{\rho}(x)) \\ &= \frac{1}{4} < [L(x), R(m(x))(df_1 + s^*(df_1))], df_2 + s^*(df_2) > -(1 \leftrightarrow 2) \\ &= \frac{1}{4} < R(m(x))(df_1 + s^*(df_1)), ad_{L(x)}^* df_2 - s^* \circ ad_{L(x)}^* df_2 > -(1 \leftrightarrow 2) \\ &= 0. \end{aligned}$$

(b) The equations of motion is a consequence of Proposition 3.11 and the fact that $\tilde{\rho}$ is Poisson. On the other hand, the assertion on ϕ_t^{red} is a corollary of Theorem 2.16 in [OR] and the relation $\tilde{\rho} \circ \phi_t \circ i_\mu = \psi_t \circ \tilde{\rho} \circ i_\mu$. \square

Case 2. The case of realization in $(\Gamma, \{\cdot, \cdot\}_R)$

Let $\mathcal{P} : X \longrightarrow \Gamma$ be a realization map of X in the coboundary dynamical Poisson groupoid $(\Gamma, \{\cdot, \cdot\}_R)$. Recall from [L1] that Γ equipped with the action

$$\mathcal{B} : H \times \Gamma \longrightarrow \Gamma, \mathcal{B}_h(u, g, v) = (Ad_{h^{-1}}^* u, hgh^{-1}, Ad_{h^{-1}}^* v) \quad (3.33)$$

is a Hamiltonian H -space with equivariant momentum map

$$\alpha - \beta : \Gamma \longrightarrow \mathfrak{h}^*, (u, g, v) \mapsto u - v. \quad (3.34)$$

We begin with the following assumption:

G1. there exists a Poisson involution

$$\Sigma : (\Gamma, \{\cdot, \cdot\}_R) \longrightarrow (\Gamma, \{\cdot, \cdot\}_R), (u, g, v) \mapsto (s_{\mathfrak{h}}^*(v), S(g), s_{\mathfrak{h}}^*(u)) \quad (3.35)$$

on Γ (where s satisfies the assumptions in Proposition 3.9) such that

$$\Sigma \circ \mathcal{P} = \mathcal{P} \circ \kappa. \quad (3.36)$$

Under G1, the map $\tilde{\mathcal{P}} = \mathcal{P} \mid X^\kappa : X^\kappa \longrightarrow \Gamma^\Sigma$ is a well-defined Poisson map by Corollary 3.6, when the stable loci are equipped with the induced structures. Let $I(G)$ be the ring of central functions in G and let $I(G^S)$ consists of restrictions of functions in $I(G)$ to the stable locus G^S of S . If \mathbf{Pr}_2 denote the projection map $\Gamma^\Sigma \longrightarrow G^S$, $(u, g, s_{\mathfrak{h}}^*(u)) \mapsto g$, a natural family of invariant functions on Γ^Σ is $\mathbf{Pr}_2^* I(G^S)$. As in the algebroid case, we begin with a special situation.

Theorem 3.18. *If $s_{\mathfrak{h}}^*(u) = u$ for all $u \in U$ so that Γ^Σ coincides with the gauge group bundle $\mathcal{I}\Gamma$ of Γ , then the functions in $\mathbf{Pr}_2^* I(G^S)$ Poisson commutes in $\Gamma^\Sigma \simeq U \times G^S$. Therefore, under the additional assumption that G1 is satisfied, $\tilde{\mathcal{P}}^* \mathbf{Pr}_2^* I(G^S)$ is a Poisson commuting family of functions on X^κ .*

Proof. Let $\varphi, \psi \in I(G)$ and let $\tilde{\varphi} = \varphi \mid G^S$, $\tilde{\psi} = \psi \mid G^S$. Then on using the first expression in (3.9), we have

$$\begin{aligned} & \{\mathbf{Pr}_2^* \tilde{\varphi}, \mathbf{Pr}_2^* \tilde{\psi}\}_{\Gamma^\Sigma}(u, g, u) \\ &= \langle R(u) D' \widetilde{\mathbf{Pr}_2^* \varphi}, D' \widetilde{\mathbf{Pr}_2^* \psi} \rangle - \langle R(u) D \widetilde{\mathbf{Pr}_2^* \varphi}, D \widetilde{\mathbf{Pr}_2^* \psi} \rangle \end{aligned}$$

where in the second line of the above formula, we have used the same symbol \mathbf{Pr}_2 to denote the projection map from Γ to G . Now, by a direct computation, we can check that

$$D' \widetilde{\mathbf{Pr}_2^* \varphi} = D \widetilde{\mathbf{Pr}_2^* \varphi} = \frac{1}{2}(D\varphi + s^*(D\varphi)).$$

Hence the two terms in the above expression cancel out. The second assertion is now clear as $\tilde{\mathcal{P}}$ is Poisson under G1. \square

In the general case when the assumption in the above theorem is not satisfied, we have

$$\begin{aligned} \{\mathbf{P}r_2^*\tilde{\varphi}, \mathbf{P}r_2^*\tilde{\psi}\}_{\Gamma^\Sigma}(u, g, s_{\mathfrak{h}}^*(u)) &= \frac{1}{4} < R(s_{\mathfrak{h}}^*(u))(D\varphi + s^*(D\varphi)), D\psi + s^*(D\psi) > \\ &\quad - \frac{1}{4} < R(u)(D\varphi + s^*(D\varphi)), D\psi + s^*(D\psi) > \end{aligned} \quad (3.37)$$

for $\varphi, \psi \in I(G)$. Therefore, $\mathbf{P}r_2^*I(G^S)$ is no longer a Poisson commuting family of functions on Γ^Σ . However, the two terms in (3.37) above do cancel out on $\Gamma^\Sigma \cap \mathcal{I}\Gamma = \{(u, g, u) \mid u \in U_s, g \in G^S\}$ where U_s is defined in (3.19). Analogous to Case 1, we now describe a situation where we can construct Poisson commuting functions on a reduced phase space. To prepare the way for Poisson reduction, we shall make the following assumptions in addition to G1:

G2. the realization map \mathcal{P} is H -equivariant,

G3. for some Lie subgroup D of H ,

$$\mathcal{B}_d \circ \Sigma = \Sigma \circ \mathcal{B}_d, \quad \mathcal{C}_d \circ \kappa = \kappa \circ \mathcal{C}_d, \quad \forall d \in D, \quad (3.38)$$

G4. $u - s_{\mathfrak{h}}^*(u) \in \mathfrak{d}^*$ for all $u \in U$, where $\mathfrak{d} = \text{Lie}(D)$.

Proposition 3.19. *Under assumptions G1-G4, the stable loci X^κ, Γ^Σ are Hamiltonian D -spaces with equivariant momentum maps $\tilde{J} = i_{\mathfrak{d}}^* \circ (J \mid X^\kappa)$ and $\tilde{\alpha} - \tilde{\beta} = \alpha - \beta \mid \Gamma^\Sigma$ respectively. Moreover, the map*

$$\tilde{\mathcal{P}} = \mathcal{P} \mid X^\kappa : X^\kappa \longrightarrow \Gamma^\Sigma \quad (3.39)$$

is a D -equivariant Poisson map.

Proof. The assertion follows from Corollaries 3.6 and 3.8, as in Proposition 3.16. \square

In order to obtain Poisson commuting functions in the general case, it is natural (in view of the remark after (3.37)) to make the following additional assumption:

G5. for some regular value $\mu \in \mathfrak{d}^*$ of \tilde{J} ,

$$\tilde{\mathcal{P}}(\tilde{J}^{-1}(\mu)) \subset (\tilde{\alpha} - \tilde{\beta})^{-1}(0) \simeq U_s \times G^S. \quad (3.40)$$

Analogous to the algebroid case, we have the Poisson variety

$$\left(\Gamma_0^\Sigma = (\tilde{\alpha} - \tilde{\beta})^{-1}(0)/D, \{\cdot, \cdot\}_{\Gamma_0^\Sigma} \right) \quad (3.41)$$

with the inclusion map $\mathbf{i}_D : (\tilde{\alpha} - \tilde{\beta})^{-1}(0) \longrightarrow \Gamma^\Sigma$ and the canonical projection $\mathbf{pr}_D : (\tilde{\alpha} - \tilde{\beta})^{-1}(0) \longrightarrow \Gamma_0^\Sigma$. Moreover, under G5, the map $\tilde{\mathcal{P}}$ in Proposition 3.19 induces a Poisson map

$$\widehat{\mathcal{P}} : X_\mu^\kappa \longrightarrow \Gamma_0^\Sigma \simeq (U_s \times G^S)/D. \quad (3.42)$$

We shall use the following notation. For $\varphi \in I(G)$, the unique function in $C^\infty(\Gamma_0^\Sigma)$ determined by $\mathbf{i}_D^* \mathbf{Pr}_2^* \tilde{\varphi}$ ($\tilde{\varphi} = \varphi|G^S$) will be denoted by $\bar{\varphi}$. Also, we set

$$\mathbf{L} = \mathbf{Pr}_2 \circ \tilde{\mathcal{P}} : X^\kappa \longrightarrow G^S, \quad (3.43)$$

$$\mathbf{m}_1 = \tilde{\alpha} \circ \tilde{\mathcal{P}} : X^\kappa \longrightarrow U, \quad (3.44)$$

and

$$\mathbf{m}_2 = \tilde{\beta} \circ \tilde{\mathcal{P}} : X^\kappa \longrightarrow U, \quad (3.45)$$

i.e. $\tilde{\mathcal{P}} = (\mathbf{m}_1, \mathbf{L}, \mathbf{m}_2)$.

Theorem 3.20. *If $\Gamma^\Sigma \neq \mathcal{IT}$, then under assumptions G1-G5, there exists a unique Poisson structure $\{\cdot, \cdot\}_{X_\mu^\kappa}$ on the reduced space $X_\mu^\kappa = \tilde{J}^{-1}(\mu)/D_\mu$ and a unique Poisson map $\widehat{\mathcal{P}} : X_\mu^\kappa \longrightarrow \Gamma_0^\Sigma$ such that*

- (a) *functions $\widehat{\mathcal{P}}^* \bar{\varphi}$, $\varphi \in I(G)$, Poisson commute in $(X_\mu^\kappa, \{\cdot, \cdot\}_{X_\mu^\kappa})$,*
- (b) *if ψ_t is the induced flow on $(\tilde{\alpha} - \tilde{\beta})^{-1}(0) \subset \mathcal{IT}$ generated by the Hamiltonian $\mathbf{Pr}_2^* \tilde{\varphi}$, $\varphi \in I(G)$ and ϕ_t is the Hamiltonian flow of $\mathbf{L}^* \tilde{\varphi}$ on X^κ , then under the flow ϕ_t , we have*

$$\begin{aligned} \frac{d}{dt} \mathbf{m}_1(\phi_t) &= \frac{1}{2} \iota^*(D\varphi + s^*(D\varphi)) \\ \frac{d}{dt} \mathbf{L}(\phi_t) &= \frac{1}{2} T_e l_{\mathbf{L}(\phi_t)} R(s_\eta^*(\mathbf{m}_1(\phi_t))(D\varphi + s^*(D\varphi)) \\ &\quad - \frac{1}{2} T_e r_{\mathbf{L}(\phi_t)} R(\mathbf{m}_1(\phi_t))(D\varphi + s^*(D\varphi)) \\ \frac{d}{dt} \mathbf{m}_2(\phi_t) &= \frac{1}{2} \iota^*(D\varphi + s^*(D\varphi)). \end{aligned}$$

Proof. (a) Let $\varphi_1, \varphi_2 \in I(G)$, then from the invariance properties of these functions, we can check that $\mathbf{Pr}_2^* \tilde{\varphi}_1, \mathbf{Pr}_2^* \tilde{\varphi}_2$ are extensions of $\mathbf{pr}_D^* \bar{\varphi}_1, \mathbf{pr}_D^* \bar{\varphi}_2$ with differentials vanishing on the tangent spaces of the D -orbits. Since $\widehat{\mathcal{P}}$ is Poisson, by making use of this fact, it follows that for $x \in \tilde{J}^{-1}(\mu)$, we have

$$\begin{aligned} &\{\widehat{\mathcal{P}}^* \bar{\varphi}_1, \widehat{\mathcal{P}}^* \bar{\varphi}_2\}_{X_\mu^\kappa} \circ \pi_\mu(x) \\ &= \{\bar{\varphi}_1, \bar{\varphi}_2\}_{\Gamma_0^\Sigma} \circ \mathbf{pr}_D(\tilde{\mathcal{P}}(x)) \\ &= \{\mathbf{Pr}_2^* \tilde{\varphi}_1, \mathbf{Pr}_2^* \tilde{\varphi}_2\}_{\Gamma^\Sigma} \circ \tilde{\mathcal{P}}(x) \\ &= 0 \end{aligned}$$

where in the last step we have invoked the formula in (3.37) and assumption G5.

(b) This follows from Proposition 3.13 and the fact that $\tilde{\mathcal{P}}$ is Poisson.

4. Spin Calogero-Moser systems associated with symmetric Lie subalgebras.

A symmetric Lie algebra is a Lie algebra equipped with a Lie algebra involution. If (\mathfrak{g}, η) is a symmetric Lie algebra, then the fixed point set \mathfrak{g}^η will be called a symmetric Lie subalgebra of \mathfrak{g} . In this section, we shall show that the general scheme in Section 3 can be applied to several examples of spin Calogero-Moser systems associated with real symmetric Lie algebras. Because the spin variables of the Dirac reduction belong to symmetric Lie subalgebras, we shall call the reduced systems spin Calogero-Moser systems associated with symmetric Lie subalgebras. In the following, we shall restrict ourselves to the trigonometric case. It will be clear that the rational case and the elliptic case can also be handled in a similar fashion and for this reason, we shall omit the details. (See Remark 4.1.12(b) and Remark 4.2.7(b) in this connection.)

4.1 Compact real forms of some spin Calogero-Moser systems.

We begin by introducing a number of Lie-theoretic objects which we will use throughout the present and the next subsections.

Let \mathfrak{g} be a complex simple Lie algebra of rank N with Killing form (\cdot, \cdot) . We fix a Cartan subalgebra \mathfrak{h} and let $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$ be the root space decomposition of \mathfrak{g} with respect to \mathfrak{h} . For each $\alpha \in \Delta$, denote by H_α the element in \mathfrak{h} which corresponds to α under the isomorphism between \mathfrak{h} and \mathfrak{h}^* induced by the Killing form (\cdot, \cdot) . On the other hand, for each $\alpha \in \Delta$, we choose root vectors $e_\alpha \in \mathfrak{g}_\alpha$ such that for all $\alpha, \beta \in \Delta$,

- (i) $[e_\alpha, e_{-\alpha}] = H_\alpha$,
- (ii) the constants $N_{\alpha, \beta}$ in the relations

$$[e_\alpha, e_\beta] = N_{\alpha, \beta} e_{\alpha+\beta}, \quad \alpha, \beta, \alpha + \beta \in \Delta$$

are real and satisfy $N_{\alpha, \beta} = -N_{-\alpha, -\beta}$.

With the notation introduced above, we define

$$\mathfrak{h}_0 = \sum_{\alpha \in \Delta} \mathbb{R} H_\alpha. \tag{4.1.1}$$

Then $(\cdot, \cdot) |_{\mathfrak{h}_0 \times \mathfrak{h}_0}$ is positive definite and each root is real-valued on \mathfrak{h}_0 . We shall fix an orthonormal basis $(x_i)_{1 \leq i \leq N}$ of \mathfrak{h}_0 in what follows.

The Lie algebra \mathfrak{g} has two standard real forms, namely, the normal real form

$$\mathfrak{g}_0 = \mathfrak{h}_0 + \sum_{\alpha \in \Delta} \mathbb{R} e_\alpha \quad (4.1.2)$$

and the compact real form

$$\mathfrak{u}_0 = i\mathfrak{h}_0 + \sum_{\alpha \in \Delta} \mathbb{R}(e_\alpha - e_{-\alpha}) + \sum_{\alpha \in \Delta} \mathbb{R}i(e_\alpha + e_{-\alpha}). \quad (4.1.3)$$

Therefore, if $\mathfrak{g}^{\mathbb{R}}$ denote the algebra \mathfrak{g} regarded as a real Lie algebra, we have $\mathfrak{g}^{\mathbb{R}} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0 = \mathfrak{u}_0 \oplus i\mathfrak{u}_0$. We shall denote the conjugation of \mathfrak{g} with respect to \mathfrak{g}_0 and \mathfrak{u}_0 by v and τ respectively. For simplicity of notation, we also write

$$v(q) = \bar{q}, \quad q \in \mathfrak{h}. \quad (4.1.4)$$

The pairing on $\mathfrak{g}^{\mathbb{R}}$ will be taken to be the Killing form on $\mathfrak{g}^{\mathbb{R}}$ scaled by the factor $\frac{1}{2}$, and will be denoted by $(\cdot, \cdot)_{\mathbb{R}}$.

In addition to the finite dimensional Lie algebras above, we will also need their corresponding loop algebras $L\mathfrak{g}$, $L\mathfrak{g}^{\mathbb{R}}$ and so on. For $X \in L\mathfrak{g}$, we shall write $X(z) = \sum_{-\infty}^{\infty} X_n z^n$ with coefficients $X_n \in \mathfrak{g}$ and similarly for the other loop algebras. Using the Killing form (\cdot, \cdot) on \mathfrak{g} , we can define a non-degenerate invariant pairing on $L\mathfrak{g}$:

$$(X, Y)_{L\mathfrak{g}} = \sum_j (X_j, Y_{-(j+1)}), \quad X, Y \in L\mathfrak{g}. \quad (4.1.5)$$

Similarly, we have a pairing on $L\mathfrak{g}^{\mathbb{R}}$, given by

$$(X, Y)_{L\mathfrak{g}^{\mathbb{R}}} = \sum_j (X_j, Y_{-(j+1)})_{\mathbb{R}}, \quad X, Y \in L\mathfrak{g}^{\mathbb{R}}. \quad (4.1.6)$$

In what follows, the connected and simply-connected Lie groups which integrate the Lie algebras $\mathfrak{g}^{\mathbb{R}}$, $\mathfrak{h}^{\mathbb{R}}$, \mathfrak{g}_0 , \mathfrak{h}_0 and $i\mathfrak{h}_0$ will be denoted by $G^{\mathbb{R}}$, $H^{\mathbb{R}}$, G_0 , H_0 and T respectively. On the other hand, U will denote a fixed connected component of $\{q \in \mathfrak{h} \mid \sin(\frac{1}{2}\alpha(q)) \neq 0 \text{ for all } \alpha \in \Delta\}$. Using the non-degeneracy of $(\cdot, \cdot)_{\mathbb{R}} |_{\mathfrak{h}^{\mathbb{R}} \times \mathfrak{h}^{\mathbb{R}}}$, $(\cdot, \cdot) |_{i\mathfrak{h}_0 \times i\mathfrak{h}_0}$ and the pairings above, we shall make the identifications $(\mathfrak{g}^{\mathbb{R}})^* \simeq \mathfrak{g}^{\mathbb{R}}$, $(\mathfrak{h}^{\mathbb{R}})^* \simeq \mathfrak{h}^{\mathbb{R}}$, $(i\mathfrak{h}_0)^* \simeq i\mathfrak{h}_0$, $\mathfrak{h}_0^* \simeq \mathfrak{h}_0$, $L\mathfrak{g}^{\mathbb{R}*} \simeq L\mathfrak{g}^{\mathbb{R}}$, where $L\mathfrak{g}^{\mathbb{R}*}$ is the restricted dual.

Consider the trigonometric dynamical r-matrix with spectral parameter (which is gauge equivalent to the one in [EV]):

$$r(q, z) = \left(c(z) + \frac{1}{12}z \right) \sum_i x_i \otimes x_i + \sum_{\alpha \in \Delta} \phi_\alpha(q, z) e^{\frac{z}{12}\alpha(q)} e_\alpha \otimes e_{-\alpha} \quad (4.1.7)$$

where

$$c(z) = \frac{1}{2} \cot \left(\frac{1}{2} z \right) \quad (4.1.8)$$

and

$$\phi_\alpha(q, z) = (c(z) + c(\alpha(q))) \quad (4.1.9)$$

Then for each $q \in U$, we can define a map $r_-^\#(q) : \mathfrak{g} \longrightarrow L\mathfrak{g}$ by the formula

$$((r_-^\#(q)\xi)(z), \eta) = (r(q, z), \eta \otimes \xi) \quad (4.1.10)$$

where $\xi, \eta \in \mathfrak{g}$. Therefore, if we write $\xi = \sum_i \xi_i x_i + \sum_{\alpha \in \Delta} \xi_\alpha e_\alpha$ for $\xi \in \mathfrak{g}$, we have explicitly that

$$(r_-^\#(q)\xi)(z) = d(z) \sum_i \xi_i x_i + \sum_{\alpha \in \Delta} \phi_\alpha(q, z) e^{\frac{z}{12}\alpha(q)} \xi_\alpha e_\alpha \quad (4.1.11)$$

where $d(z) = c(z) + \frac{1}{12}z$. We can also construct the associated classical dynamical r -matrix $R : U \longrightarrow L(L\mathfrak{g}, L\mathfrak{g})$ for the pair $(L\mathfrak{g}, \mathfrak{h})$. To do so, we will need to use the following formula:

$$\begin{aligned} & \frac{\partial^k r}{\partial z^k}(q, z) \\ &= d^{(k)}(z) \sum_i x_i \otimes x_i + \sum_{\alpha \in \Delta} \sum_{j=0}^k \binom{k}{j} \phi_\alpha^{(k-j)}(q, z) \left(\frac{1}{12} \alpha(q) \right)^j e^{\frac{z}{12}\alpha(q)} e_\alpha \otimes e_{-\alpha}. \end{aligned} \quad (4.1.12)$$

Proposition 4.1.1. *The classical dynamical r -matrix R associated with the trigonometric dynamical r -matrix with spectral parameter in (4.1.7) is given by*

$$\begin{aligned} & (R(q)X)(z) \\ &= \frac{1}{2} X(z) + \sum_{k=0}^{\infty} \frac{d^{(k)}(-z)}{k!} \Pi_{\mathfrak{h}} X_{-(k+1)} \\ &+ \sum_{k \geq 0} \frac{1}{k!} \sum_{\alpha \in \Delta} \sum_{j=0}^k \binom{k}{j} \phi_{-\alpha}^{(k-j)}(q, -z) \left(\frac{1}{12} \alpha(q) \right)^j e^{\frac{z}{12}\alpha(q)} (X_{-(k+1)})_\alpha e_\alpha. \end{aligned} \quad (4.1.13)$$

Proof. This follows upon substituting the expression for $\frac{\partial^k r}{\partial z^k}(q, z)$ in (4.1.12) into the formula for $R(q)X$ in Theorem 2.4(b). \square

Clearly, R induces a map $U \longrightarrow L(L\mathfrak{g}^{\mathbb{R}}, L\mathfrak{g}^{\mathbb{R}})$ which we shall also denote by R .

Proposition 4.1.2. *The map $R : U \longrightarrow L(L\mathfrak{g}^{\mathbb{R}}, L\mathfrak{g}^{\mathbb{R}})$ is a solution of the mDYBE for the pair $(L\mathfrak{g}^{\mathbb{R}}, \mathfrak{h}^{\mathbb{R}})$ with $c = -\frac{1}{4}$. Moreover, for $q \in U$, $\xi \in \mathfrak{g}^{\mathbb{R}}$, we have $r_-^{\#}(q)\xi \in L\mathfrak{g}^{\mathbb{R}}$.*

Proof. For $q \in U$, $X, Y \in L\mathfrak{g}$, the term $(dR(q)(\cdot)X, Y)_{L\mathfrak{g}}$ which appears in the mDYBE for the pair $(L\mathfrak{g}, \mathfrak{h})$ is the unique element in \mathfrak{h} whose pairing with $Z \in \mathfrak{h}$ is given by $(dR(q)(Z)X, Y)_{L\mathfrak{g}}$. On the other hand, the term $(dR(q)(\cdot)X, Y)_{L\mathfrak{g}^{\mathbb{R}}} \in \mathfrak{h}^{\mathbb{R}}$ has a similar meaning. But now it is easy to show from the non-degeneracy of $(\cdot, \cdot)_{\mathfrak{h}^{\mathbb{R}} \times \mathfrak{h}^{\mathbb{R}}}$ that $(dR(q)(\cdot)X, Y)_{L\mathfrak{g}} = (dR(q)(\cdot)X, Y)_{L\mathfrak{g}^{\mathbb{R}}}$. Hence it follows from this argument and Theorem 2.4(a) that the map $R : U \longrightarrow L(L\mathfrak{g}^{\mathbb{R}}, L\mathfrak{g}^{\mathbb{R}})$ is a solution of the mDYBE for the pair $(L\mathfrak{g}^{\mathbb{R}}, \mathfrak{h}^{\mathbb{R}})$. The assertion involving $r_-^{\#}(q)\xi$ is trivial. \square

We now introduce the trivial Lie groupoids

$$\Omega = U \times G^{\mathbb{R}} \times U, \quad \Gamma = U \times LG^{\mathbb{R}} \times U. \quad (4.1.14)$$

By Proposition 4.1.2 and (2.9), we can use the map $R : U \longrightarrow L(L\mathfrak{g}^{\mathbb{R}}, L\mathfrak{g}^{\mathbb{R}})$ to construct the associated coboundary dynamical Lie algebroid $A^*\Gamma \simeq U \times \mathfrak{h}^{\mathbb{R}} \times L\mathfrak{g}^{\mathbb{R}}$. Hence its dual bundle $A\Gamma \simeq U \times \mathfrak{h}^{\mathbb{R}} \times L\mathfrak{g}^{\mathbb{R}}$ has a Lie-Poisson structure. On the other hand, we shall equip the dual bundle $A^*\Omega \simeq U \times \mathfrak{h}^{\mathbb{R}} \times \mathfrak{g}^{\mathbb{R}}$ of the trivial Lie algebroid $A\Omega$ with the corresponding Lie-Poisson structure. The Poisson manifolds $A^*\Omega$ and $A\Gamma$ are Hamiltonian $H^{\mathbb{R}}$ -spaces with actions

$$\mathcal{C}_h(q, p, \xi) = (q, p, Ad_h \xi), \quad h \in H^{\mathbb{R}}, (q, p, \xi) \in A^*\Omega \quad (4.1.15)$$

and

$$\mathcal{A}_h(q, p, X) = (q, p, Ad_h X), \quad h \in H^{\mathbb{R}}, (q, p, X) \in A\Gamma \quad (4.1.16)$$

and the corresponding equivariant momentum maps are respectively given by

$$J : A^*\Omega \longrightarrow \mathfrak{h}^{\mathbb{R}}, \quad (q, p, \xi) \mapsto \Pi_{\mathfrak{h}^{\mathbb{R}}} \xi \quad (4.1.17)$$

and

$$\gamma : A\Gamma \longrightarrow \mathfrak{h}^{\mathbb{R}}, \quad (q, p, X) \mapsto p \quad (4.1.18)$$

where $\Pi_{\mathfrak{h}^{\mathbb{R}}}$ is the projection map to $\mathfrak{h}^{\mathbb{R}}$ relative to the splitting $\mathfrak{g}^{\mathbb{R}} = \mathfrak{h}^{\mathbb{R}} \oplus (\mathfrak{h}^{\mathbb{R}})^{\perp}$.

In view of our discussion above, the following result is just a real analog of what we have in Theorem 2.5.

Proposition 4.1.3. *The map*

$$\rho : A^*\Omega \longrightarrow A\Gamma, \quad (q, -\Pi_{\mathfrak{h}^\mathbb{R}}\xi, p + r_-^\#(q)\xi) \quad (4.1.19)$$

is an $H^\mathbb{R}$ -equivariant Poisson map.

The trigonometric spin Calogero-Moser system which we consider for our purpose here is the Hamiltonian system on $A^*\Omega$ generated by the Hamiltonian

$$\mathcal{H}(q, p, \xi) = \operatorname{Re} \left\{ \frac{1}{2} \sum_i p_i^2 - \frac{1}{8} \sum_{\alpha \in \Delta} \left(\frac{1}{\sin^2 \frac{1}{2}\alpha(q)} - \frac{1}{3} \right) \xi_\alpha \xi_{-\alpha} \right\}. \quad (4.1.20)$$

Let Q be the quadratic function

$$Q(X) = \frac{1}{2} \operatorname{Re} \oint_C (X(z), X(z)) \frac{dz}{2\pi iz}, \quad X \in L\mathfrak{g}^\mathbb{R} \quad (4.1.21)$$

where C is a small circle around the origin with the positive orientation.

The relation between \mathcal{H} and Q is given in our next result which is obtained by a simple residue calculation.

Proposition 4.1.4. *$\mathcal{H} = \rho^* Pr_3^* Q$, where Pr_3 is the projection map onto the third factor of $A\Gamma$. Thus the Hamiltonian system generated by \mathcal{H} can be realized in $A\Gamma$.*

We next examine the phase space underlying \mathcal{H} .

Proposition 4.1.5. *The map*

$$\kappa : A^*\Omega \longrightarrow A^*\Omega, \quad (q, p, \xi) \mapsto (\bar{q}, \bar{p}, \tau(\xi)) \quad (4.1.22)$$

is a Poisson involution with stable locus

$$A^*\Omega^\kappa = (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times \mathfrak{u}_0. \quad (4.1.23)$$

Proof. According to Proposition 3.14, we have to check that τ^* is a Lie algebra morphism. But from the orthogonality of \mathfrak{u}_0 and $i\mathfrak{u}_0$ under $(\cdot, \cdot)_\mathbb{R}$, we can show that $\tau^* = \tau$. As τ is a Cartan involution, the assertion follows. \square

As can be easily verified, the real Hamiltonian system generated by \mathcal{H} has the same equations of motion as the one on $U \times \mathfrak{h} \times \mathfrak{g}$ with (complex holomorphic) Hamiltonian

$$\mathcal{H}^\mathbb{C}(q, p, \xi) = \frac{1}{2} \sum_i p_i^2 - \frac{1}{8} \sum_{\alpha \in \Delta} \left(\frac{1}{\sin^2 \frac{1}{2}\alpha(q)} - \frac{1}{3} \right) \xi_\alpha \xi_{-\alpha}. \quad (4.1.24)$$

This latter Hamiltonian system, on the other hand, has a *compact real form*, corresponding to $q_i, p_i \in \mathbb{R}, i = 1, \dots, N$ and $\xi_\alpha = -\bar{\xi}_{-\alpha}, \alpha \in \Delta$. More precisely, the compact real form of \mathcal{H}^C is the Hamiltonian system on $A^*\Omega^\kappa$ generated by

$$\begin{aligned}\tilde{\mathcal{H}}(q, p, \xi) &= (\mathcal{H} \mid A^*\Omega^\kappa)(q, p, \xi) \\ &= \frac{1}{2} \sum_i p_i^2 + \frac{1}{8} \sum_{\alpha \in \Delta} \left(\frac{1}{\sin^2 \frac{1}{2}\alpha(q)} - \frac{1}{3} \right) |\xi_\alpha|^2.\end{aligned}\quad (4.1.25)$$

In the rest of the subsection, we shall consider the realization of this compact real form and its associated integrable model. To start with, we introduce the map

$$s : L\mathfrak{g}^{\mathbb{R}} \longrightarrow L\mathfrak{g}^{\mathbb{R}}, \quad s(X)(z) = -\tau(X(-\bar{z})) = -\sum_j \tau(X_j)(-z)^j. \quad (4.1.26)$$

Proposition 4.1.6. *The map*

$$\sigma : A\Gamma \longrightarrow A\Gamma, \quad (q, p, X) \mapsto (\bar{q}, -\bar{p}, s(X)) \quad (4.1.27)$$

is a Poisson involution with stable locus

$$A\Gamma^\sigma = (U \cap \mathfrak{h}_0) \times i\mathfrak{h}_0 \times (L\mathfrak{g}^{\mathbb{R}})^s \quad (4.1.28)$$

where

$$(L\mathfrak{g}^{\mathbb{R}})^s = \{X \in L\mathfrak{g}^{\mathbb{R}} \mid X_{2j+1} \in \mathfrak{u}_0, X_{2j} \in i\mathfrak{u}_0 \text{ for all } j\}. \quad (4.1.29)$$

Hence $(A^*\Gamma, \sigma^*)$ is a symmetric coboundary dynamical Lie algebroid.

Proof. It is clear from (4.1.19) that s is an involutive Lie algebra anti-morphism. If $Z \in \mathfrak{h}^{\mathbb{R}}$, we have $\tau(Z) = -\bar{Z}$ from which it follows that $s_{\mathfrak{h}^{\mathbb{R}}}(Z) = \bar{Z}$. Next, we show that $s^* = -s$. To do so, take $X, Y \in L\mathfrak{g}^{\mathbb{R}}$, then

$$\begin{aligned}& (s^*(X), Y)_{L\mathfrak{g}^{\mathbb{R}}} \\ &= \sum_j (X_j, -\tau(Y_{-(j+1)})(-1)^{j+1})_{\mathbb{R}} \\ &= \sum_j ((-1)^j \tau(X_j), Y_{-(j+1)})_{\mathbb{R}} \\ &= \left(\sum_j \tau(X_j)(-z)^j, Y \right)_{L\mathfrak{g}^{\mathbb{R}}} \\ &= (-s(X), Y)_{L\mathfrak{g}^{\mathbb{R}}},\end{aligned}$$

as required. We are now ready to compute $s \circ R(q) \circ s^*$ for $q \in U$. Instead of using the explicit expression in (4.1.13), we will do this using the relationship between R and r . This is more illuminating as the property of R should follow from that of r . To start with, we have

$$\begin{aligned} & (R(q)s(X))(z) \\ &= \frac{1}{2}s(X)(z) + \sum_{k \geq 0} \frac{1}{k!} \left(\frac{\partial^k r}{\partial z^k}(q, -z), (-1)^k \tau(X_{-(k+1)}) \otimes 1 \right). \end{aligned}$$

Therefore,

$$\begin{aligned} & (s \circ R(q) \circ s^*)(X)(z) \\ &= \tau(R(q)s(X)(-\bar{z})) \\ &= -\frac{1}{2}X(z) + \sum_{k \geq 0} \frac{1}{k!} \tau \left(\frac{\partial^k r}{\partial z^k}(q, \bar{z}), (-1)^k \tau(X_{-(k+1)}) \otimes 1 \right). \end{aligned}$$

To simplify the above expression, note that $\overline{(a, \tau(\xi))} = (\tau(a), \xi)$ for all $a, \xi \in \mathfrak{g}$. From this relation, we find

$$\tau(a \otimes b, \tau(\xi) \otimes 1) = (\tau^{\otimes 2}(a \otimes b), \xi \otimes 1)$$

for all $a, b, \xi \in \mathfrak{g}$. Consequently,

$$\begin{aligned} & \tau \left(\frac{\partial^k r}{\partial z^k}(q, \bar{z}), (-1)^k \tau(X_{-(k+1)}) \otimes 1 \right) \\ &= \left(\tau^{\otimes 2} \left(\frac{\partial^k r}{\partial z^k}(q, \bar{z}) \right), (-1)^k X_{-(k+1)} \otimes 1 \right). \end{aligned}$$

But from (4.1.12), we can verify that

$$\tau^{\otimes 2} \left(\frac{\partial^k r}{\partial z^k}(q, \bar{z}) \right) = -(-1)^k \frac{\partial^k r}{\partial z^k}(\bar{q}, -z).$$

Substitute this into the above expression, we obtain

$$\begin{aligned} & \tau \left(\frac{\partial^k r}{\partial z^k}(q, \bar{z}), (-1)^k \tau(X_{-(k+1)}) \otimes 1 \right) \\ &= - \left(\frac{\partial^k r}{\partial z^k}(\bar{q}, -z), X_{-(k+1)} \otimes 1 \right) \end{aligned}$$

and hence that

$$(s \circ R(q) \circ s^*)(X)(z) = -(R(\bar{q})X)(z).$$

Therefore, we can now conclude from Proposition 3.9 that the map σ is a Poisson involution. \square

Proposition 4.1.7. (a) $\sigma \circ \rho = \rho \circ \kappa$.

(b) For all $d \in T$,

$$\mathcal{A}_d \circ \sigma = \sigma \circ \mathcal{A}_d, \quad \mathcal{C}_d \circ \kappa = \kappa \circ \mathcal{C}_d.$$

(c) $\{Z \in \mathfrak{h}^{\mathbb{R}} \mid s_{\mathfrak{h}^{\mathbb{R}}}^*(Z) = -Z\} = \mathfrak{t}$.

Proof. (a) For any $(q, p, \xi) \in A^*\Omega$, we have

$$\rho \circ \kappa(q, p, \xi) = (\bar{q}, -\Pi_{\mathfrak{h}^{\mathbb{R}}} \tau(\xi), \bar{p} + r_-^\sharp(\bar{q})\tau(\xi)).$$

On the other hand,

$$\sigma \circ \rho(q, p, \xi) = (\bar{q}, \overline{\Pi_{\mathfrak{h}^{\mathbb{R}}} \xi}, \bar{p} + s(r_-^\sharp(q)\xi)).$$

Now, from the fact that τ preserves $\mathfrak{h}^{\mathbb{R}}$, we have

$$\Pi_{\mathfrak{h}^{\mathbb{R}}} \tau(\xi) = -\overline{\Pi_{\mathfrak{h}^{\mathbb{R}}} \xi}.$$

Therefore, it remains to show that $s(r_-^\sharp(q)\xi) = r_-^\sharp(\bar{q})\tau(\xi)$. To do so, we invoke the explicit expression for $r_-^\sharp(q)(\xi)$, according to which

$$\begin{aligned} & (r_-^\sharp(\bar{q})\tau(\xi))(z) \\ &= d(z) \sum_i \tau(\xi)_i x_i + \sum_{\alpha \in \Delta} \phi_\alpha(\bar{q}, z) e^{\frac{\bar{z}}{12} \alpha(\bar{q})} (\tau(\xi))_\alpha e_\alpha \\ &= -d(z) \sum_i \bar{\xi}_i x_i - \sum_{\alpha \in \Delta} \phi_\alpha(\bar{q}, z) e^{\frac{\bar{z}}{12} \alpha(\bar{q})} \bar{\xi}_{-\alpha} e_\alpha. \end{aligned}$$

But on the other hand,

$$\begin{aligned} & s(r_-^\sharp(q)\xi)(z) \\ &= -d(-z) \sum_i \bar{\xi}_i \tau(x_i) - \sum_{\alpha \in \Delta} \phi_\alpha(\bar{q}, -z) e^{-\frac{\bar{z}}{12} \alpha(\bar{q})} \bar{\xi}_\alpha \tau(e_\alpha) \\ &= -d(z) \sum_i \bar{\xi}_i x_i - \sum_{\alpha \in \Delta} \phi_\alpha(\bar{q}, z) e^{\frac{\bar{z}}{12} \alpha(\bar{q})} \bar{\xi}_{-\alpha} e_\alpha \end{aligned}$$

and so we have equality.

(b) From the definition of κ and \mathcal{C} , it is easy to see that $\mathcal{C}_d \circ \kappa = \kappa \circ \mathcal{C}_d$ for $d \in T$ if and only if $\tau(Ad_d \xi) = Ad_d \tau(\xi)$ for $d \in T$. But the latter is clear as $\tau = 1$ on \mathfrak{u}_0 and $i\mathfrak{h}_0 \subset \mathfrak{u}_0$. The validity of the other assertion also boils down to the same condition above, as can be verified from the definition of s .

(c) From the proof of Proposition 4.1.6, we have $s_{\mathfrak{h}^{\mathbb{R}}}(Z) = \overline{Z}$ for $Z \in \mathfrak{h}^{\mathbb{R}}$ from which the assertion clearly follows. \square

Combining Proposition 4.1.3 and Proposition 4.1.7, we conclude that assumptions A1-A4 are satisfied. Hence Proposition 3.16 and the explicit form of ρ give the following result.

Corollary 4.1.8. *The stable loci $A^*\Omega^\kappa$, $A\Gamma^\sigma$ are Hamiltonian T -spaces with equivariant momentum maps $\tilde{J} : A^*\Omega^\kappa \longrightarrow \mathfrak{t}$, $(q, p, \xi) \mapsto -\Pi_{\mathfrak{h}^\mathbb{R}}\xi$, and $\tilde{\gamma} = \gamma \mid A\Gamma^\kappa$ respectively. Moreover, the map*

$$\tilde{\rho} = \rho \mid A^*\Omega^\kappa : (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times \mathfrak{u}_0 \longrightarrow (U \cap \mathfrak{h}_0) \times i\mathfrak{h}_0 \times (L\mathfrak{g}^\mathbb{R})^s \quad (4.1.30)$$

is a T -equivariant Poisson map and $\tilde{\rho}(\tilde{J}^{-1}(0)) \subset \tilde{\gamma}^{-1}(0)$.

Remark 4.1.9. At this juncture, it is tempting to reduce $\tilde{\rho}$ further to the map in (4.2.5) in the next subsection. However, it is not hard to show that this cannot be achieved as we cannot find appropriate Poisson involutions which commute with $\tilde{\rho}$.

We now introduce $L = Pr_3 \circ \tilde{\rho}$, as in (3.30). Also, let $\tilde{Q} = Q \mid (L\mathfrak{g}^\mathbb{R})^s$.

Proposition 4.1.10. (a) $\tilde{\mathcal{H}} = L^*\tilde{Q}$ and is invariant under the canonical T -action on $A^*\Omega^\kappa$.

(b) *The restriction of the Hamiltonian equations of motion generated by $\tilde{\mathcal{H}}$ on $A^*\Omega^\kappa$ to the invariant submanifold $\tilde{J}^{-1}(0)$ are given by*

$$\begin{aligned} \dot{q} &= p, \\ \dot{p} &= \frac{1}{8} \sum_{\alpha \in \Delta} \frac{\cot \frac{1}{2}\alpha(q)}{\sin^2 \frac{1}{2}\alpha(q)} |\xi_\alpha|^2 H_\alpha, \\ \dot{\xi} &= \left[\xi, -\frac{1}{4} \sum_{\alpha \in \Delta} \frac{\xi_\alpha}{\sin^2 \frac{1}{2}\alpha(q)} e_\alpha \right]. \end{aligned} \quad (4.1.30)$$

Moreover, under the Hamiltonian flow, we have

$$\dot{L}(q, p, \xi) = [L(q, p, \xi), R(q)M(q, p, \xi)] \quad (4.1.31)$$

on the invariant submanifold $\tilde{J}^{-1}(0)$ where

$$M(q, p, \xi)(z) = L(q, p, \xi)(z)/z. \quad (4.1.32)$$

Proof. (a) The assertion is clear from Proposition 4.1.4.

(b) From the expression for the Poisson bracket in (3.13), the Hamiltonian equations of motion are given by $\dot{q} = \frac{1}{2}(\delta_2 \mathcal{H} + \overline{\delta_2 \mathcal{H}})$, $\dot{p} = -\frac{1}{2}(\delta_1 \mathcal{H} + \overline{\delta_1 \mathcal{H}})$, and $\dot{\xi} = [\xi, \frac{1}{2}(\delta \mathcal{H} + \tau(\delta \mathcal{H})]$. Therefore, (4.1.30) follows by a direct computation. On the other hand, (4.1.31) is a consequence of the last equation in Theorem 3.17(b). \square

We now restrict ourselves to a smooth component of $\tilde{J}^{-1}(0)/T = (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times (\mathfrak{u}_0 \cap \mathfrak{h}^\perp)/T$. For this purpose, we introduce the following open submanifold of \mathfrak{u}_0 :

$$\mathcal{U} = \{ \xi \in \mathfrak{u}_0 \mid \xi_{\alpha_i} = (\xi, e_{-\alpha_i}) \neq 0, \quad i = 1, \dots, N \}. \quad (4.1.33)$$

Clearly, \mathcal{U} is dense in \mathfrak{u}_0 and is stable under the T -action. Therefore, $(U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times \mathcal{U}$ is a Poisson submanifold of $A^*\Omega^\kappa$, and we can check that the T -action on $A^*\Omega^\kappa$ induces a locally free Hamiltonian group action on $(U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times \mathcal{U}$. Consequently, the corresponding momentum map is given by the restriction of the one in Corollary 4.1.8. To simplify notation, we shall denote this momentum map also by \tilde{J} so that $\tilde{J}^{-1}(0) = (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times (\mathcal{U} \cap \mathfrak{h}^\perp)$. Now observe that under the T -action, all the isotropy subgroups of the elements of $\tilde{J}^{-1}(0)$ are identical. Since T is compact, it follows from the above observation that the orbit space $\tilde{J}^{-1}(0)/T = (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times (\mathcal{U} \cap \mathfrak{h}^\perp/T)$ is a smooth manifold. We shall fix a branch of the argument function. Then the formula

$$h(\xi) = \exp \left(i \sum_{j,k=1}^N C_{kj} \arg \xi_{\alpha_k} h_{\alpha_j} \right) \quad (4.1.34)$$

defines a map $h : \mathcal{U} \cap \mathfrak{h}^\perp \longrightarrow T$ where $C = (C_{jk})$ is the inverse of the Cartan matrix and $h_{\alpha_j} = \frac{2}{(\alpha_j, \alpha_j)} H_{\alpha_j}$, $j = 1, \dots, N$. Note that for $\xi \in \mathcal{U} \cap \mathfrak{h}^\perp$, the element $Ad_{h(\xi)}^{-1} \xi$ is such that $(Ad_{h(\xi)}^{-1} \xi, e_{-\alpha_i}) > 0$, $i = 1, \dots, N$. We shall henceforth identify the reduced phase space $\tilde{J}^{-1}(0)/T = (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times (\mathcal{U} \cap \mathfrak{h}^\perp/T)$ with $(U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times (\mathfrak{u}_0)_{red}$ under the identification map $(q, p, [\xi]) \mapsto (q, p, Ad_{h(\xi)}^{-1} \xi)$ where

$$(\mathfrak{u}_0)_{red} = \{ f \in \mathcal{U} \cap \mathfrak{h}^\perp \mid f_{\alpha_i} = (f, e_{-\alpha_i}) > 0, \quad i = 1, \dots, N \}. \quad (4.1.35)$$

By Poisson reduction, the reduced manifold $(U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times (\mathfrak{u}_0)_{red}$ has a unique Poisson structure where the factor $(\mathfrak{u}_0)_{red}$ is equipped with the reduction (at 0) of the Lie-Poisson structure on \mathcal{U} by the T -action. Moreover, the reduction of the Hamiltonian $\tilde{\mathcal{H}}$ on $A^*\Omega^\kappa$ is given by

$$\tilde{\mathcal{H}}_0(q, p, f) = \frac{1}{2} \sum_i p_i^2 + \frac{1}{8} \sum_{\alpha \in \Delta} \left(\frac{1}{\sin^2 \frac{1}{2} \alpha(q)} - \frac{1}{3} \right) |f_\alpha|^2. \quad (4.1.36)$$

Combining Corollary 4.1.8, Proposition 4.1.10, Theorem 3.17 together with our discussion above, we therefore obtain the following result. (See the discussion preceding (3.32) for notations.)

Theorem 4.1.11. *The map $\tilde{\rho}$ in (4.1.30) restricted to $(U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times \mathcal{U}$ induces a unique Poisson map $\hat{\rho}: (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times (\mathfrak{u}_0)_{red} \longrightarrow A\Gamma_0^\sigma$ such that*

- (a) *functions $\mathcal{F}_0 = \hat{\rho}^* \bar{f}$, $f \in I(L\mathfrak{g})$, Poisson commute in $(U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times (\mathfrak{u}_0)_{red}$ and provide a family of conserved quantities in involution for the Hamiltonian $\tilde{\mathcal{H}}_0$,*
 (b) *Under the Hamiltonian flow generated by $\tilde{\mathcal{H}}_0$ on $(U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times (\mathfrak{u}_0)_{red}$,*

$$\dot{L}(q, p, f) = [L(q, p, f), R(q)M(q, p, f) + \mathcal{M}]$$

where

$$\mathcal{M} = \frac{i}{4} \sum_{j,k} \frac{C_{kj}}{f_{\alpha_k}} \sum_{\substack{\alpha \in \Delta \\ \alpha_k - \alpha \in \Delta}} N_{\alpha, \alpha_k - \alpha} \frac{\text{Im}(f_\alpha f_{\alpha_k - \alpha})}{\sin^2 \frac{1}{2} \alpha(q)} h_{\alpha_j}.$$

Proof. Only (b) requires a proof. First of all, from (4.1.31) and the T -equivariance of $\tilde{\rho}$ and R , we have (for $f = \text{Ad}_{h(\xi)^{-1}} \xi$):

$$\begin{aligned} & \dot{L}(q, p, f) \\ &= \left[L(q, p, f), R(q)M(q, p, f) + T_{h(\xi)} l_{h(\xi)^{-1}} \frac{d}{dt} h(\xi) \right]. \end{aligned}$$

Now, differentiating $h(\xi)$ yields

$$T_{h(\xi)} l_{h(\xi)^{-1}} \frac{d}{dt} h(\xi) = i \sum_{j,k} C_{kj} (\arg \xi_{\alpha_k})^\cdot h_{\alpha_j}. \quad (*)$$

But

$$\begin{aligned} \dot{\xi}_{\alpha_k} &= \left(\left[\xi, -\frac{1}{4} \sum_{\alpha \in \Delta} \frac{\xi_\alpha}{\sin^2 \frac{1}{2} \alpha(q)} e_\alpha \right], e_{-\alpha_k} \right) \\ &= \frac{1}{4} \sum_{\substack{\alpha \in \Delta \\ \alpha_k - \alpha \in \Delta}} N_{\alpha, \alpha_k - \alpha} \frac{\xi_\alpha \xi_{\alpha_k - \alpha}}{\sin^2 \frac{1}{2} \alpha(q)}. \end{aligned}$$

Therefore, upon dividing both sides of the above expression by $e^{i \arg \xi_{\alpha_k}}$ and taking the imaginary part of both sides, we find

$$f_{\alpha_k} (\arg \xi_{\alpha_k})^\cdot = \frac{1}{4} \sum_{\substack{\alpha \in \Delta \\ \alpha_k - \alpha \in \Delta}} N_{\alpha, \alpha_k - \alpha} \frac{\text{Im}(f_\alpha f_{\alpha_k - \alpha})}{\sin^2 \frac{1}{2} \alpha(q)}$$

where we have used the reality of $N_{\alpha, \beta}$ and $\alpha(q)$ together with the fact that $f = \sum_{\alpha \in \Delta} \xi_\alpha e^{-i \sum_k m_\alpha^k \arg \xi_{\alpha_k}} e_\alpha$ (the m_α^k are defined by $\alpha = \sum_k m_\alpha^k e_{\alpha_k}$). Consequently, when we substitute this in (*), the desired expression for \mathcal{M} follows. \square

Remark 4.1.12. (a) The Hamiltonian $\tilde{\mathcal{H}}_0$ is in fact completely integrable in the sense of Liouville on generic symplectic leaves of the reduced phase space. The same remark also applies to the integrable spin Calogero-Moser systems in [LX2] for all simple Lie algebras. A unifying and representation independent method to establish the Liouville integrability of such systems for all simple Lie algebras will be given in a forthcoming paper. For $\mathfrak{g} = gl(N, \mathbb{C})$ with \mathfrak{h} taken to be the set of diagonal matrices in \mathfrak{g} , a sketch of the proof will be given below.

(b) For the rational dynamical r-matrix $\frac{\Omega}{z} + \sum_{\alpha \in \Delta} \frac{1}{\alpha(q)} e_{\alpha} \otimes e_{-\alpha}$ and the elliptic dynamical r-matrix $\zeta(z) \sum_i x_i \otimes x_i - \sum_{\alpha \in \Delta} l(\alpha(q), z) e_{\alpha} \otimes e_{-\alpha}$ (here $l(w, z) = -\frac{\sigma(w+z)}{\sigma(w)\sigma(z)}$), recall that we can associate the corresponding (complex holomorphic) spin Calogero-Moser systems [LX2]. We remark that the compact real forms of these Hamiltonian systems can also be treated in the same way. Indeed, with the corresponding $r_{-}^{\sharp}(q)$ and $R(q)$, our analysis above can be repeated and everything goes through just the same as before. Note that the explicit form of $r(q, z)$ is only used in checking

$$\tau^{\otimes 2} \left(\frac{\partial^k r}{\partial z^k}(q, \bar{z}) \right) = -(-1)^k \frac{\partial^k r}{\partial z^k}(\bar{q}, -z)$$

and in verifying $s(r_{-}^{\sharp}(q)\xi) = r_{-}^{\sharp}(\bar{q})\tau(\xi)$. Finally we remark that a version of the rational spin Calogero-Moser system similar to the compact real form of our rational case has been obtained in [AKLM] by reducing a free Hamiltonian system on a cotangent bundle. However, it is not clear how this method can be generalized to handle the elliptic case.

In the remainder of the subsection, we shall give a brief sketch of the Liouville integrability for the reductive case where $\mathfrak{g} = gl(N, \mathbb{C})$. Indeed, it is easy to see that we can repeat the same analysis above for this case with \mathfrak{h} taken to be the set of all diagonal matrices in \mathfrak{g} and with the trigonometric dynamical r-matrix with spectral parameter

$$r(q, z) = \left(c(z) + \frac{1}{12}z \right) \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} (c(z) + c(q_i - q_j)) e^{\frac{z}{12}(q_i - q_j)} e_{ij} \otimes e_{ji}$$

where e_{ij} is the $N \times N$ matrix with a 1 in the (i, j) entry and zeros elsewhere and $c(z)$ is as in (4.1.8). In this case, \mathfrak{h}_0 and \mathfrak{u}_0 are the subalgebras of $\mathfrak{g}^{\mathbb{R}}$ consisting of real diagonal matrices and skew-Hermitian matrices respectively and we take U to be a fixed connected component of $\{q \in \mathfrak{h} \mid \sin(\frac{q_i - q_j}{2}) \neq 0 \text{ for all } i \neq j\}$. Moreover, $\mathfrak{g}_0 = gl(N, \mathbb{R})$ and T is the maximal torus of the unitary group $U(N)$ consisting of unitary diagonal matrices. For our analysis below, we also have to introduce

the torus $T' \subset T$ consisting of matrices of the form $\text{diag}(1, e^{i\theta_2}, \dots, e^{i\theta_N})$. Clearly, the above results for simple Lie algebras have obvious analogs in this case. In particular, we can obtain the Hamiltonian

$$\tilde{\mathcal{H}}(q, p, \xi) = \frac{1}{2} \sum_i p_i^2 + \frac{1}{8} \sum_{i \neq j} \left(\frac{1}{\sin^2(\frac{q_i - q_j}{2})} - \frac{1}{3} \right) |\xi_{ij}|^2$$

on $A^*\Omega^\kappa = (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times \mathfrak{u}_0$ and its associated realization map

$$\tilde{\rho}: (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times \mathfrak{u}_0 \longrightarrow (U \cap \mathfrak{h}_0) \times i\mathfrak{h}_0 \times (L\mathfrak{g}^\mathbb{R})^s$$

by Dirac reduction. For the Hamiltonian system generated by $\tilde{\mathcal{H}}$, we note that the restriction of its equations of motion to the invariant submanifold $\tilde{\mathcal{J}}^{-1}(0)$ are given by

$$\begin{aligned} \dot{q} &= p, \\ \dot{p} &= \frac{1}{4} \sum_{i \neq j} \frac{\cot(\frac{q_i - q_j}{2})}{\sin^2(\frac{q_i - q_j}{2})} |\xi_{ij}|^2 e_{ii}, \\ \dot{\xi} &= \left[\xi, -\frac{1}{4} \sum_{i \neq j} \frac{\xi_{ij}}{\sin^2(\frac{q_i - q_j}{2})} e_{ij} \right]. \end{aligned} \tag{4.1.37}$$

Thus the equations coincide exactly with the ones derived in [H] (cf. also [HKS] and [NM]) for the eigenphases and (essentially) the eigenvectors of the unitary Floquet operator $F = e^{-i\lambda V} e^{-iH_0}$ (as a function of λ) associated with a periodically kicked quantum system if we take the time variable in (4.1.37) to be the kick strength λ . More importantly, we have the Lax operator $L = Pr_3 \circ \tilde{\rho}$ whose restriction to $\tilde{\mathcal{J}}^{-1}(0) = (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times (\mathfrak{u}_0 \cap \mathfrak{h}^\perp)$ is given explicitly by

$$L(q, p, \xi)(z) = p + \sum_{i \neq j} (c(z) + c(q_i - q_j)) e^{\frac{z}{12}(q_i - q_j)} \xi_{ij} e_{ij} \tag{4.1.38}$$

and we can establish Liouville integrability for the reduced Hamiltonian system using L . To do so, we introduce the open submanifold of \mathfrak{u}_0 :

$$\mathcal{U} = \{ \xi \in \mathfrak{u}_0 \mid \xi_{i, i+1} \neq 0, \quad i = 1, \dots, N \}. \tag{4.1.39}$$

Then \mathcal{U} is dense in \mathfrak{u}_0 and is stable under the T -action. Hence $(U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times \mathcal{U}$ is a Poisson submanifold of $A^*\Omega^\kappa$ and the T -action on $A^*\Omega^\kappa$ induces a Hamiltonian group action on $(U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times \mathcal{U}$. Denote the momentum map of this action also by $\tilde{\mathcal{J}}$. Then $\tilde{\mathcal{J}}^{-1}(0) = (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times (\mathcal{U} \cap \mathfrak{h}^\perp)$. Since the isotropy subgroups of the elements of $\tilde{\mathcal{J}}$ under the T -action are all equal to the center of $U(N)$ ($= \{e^{i\theta} I\} \simeq S^1$) and T

is compact, we conclude that the orbit space $\tilde{J}^{-1}(0)/T = (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times (\mathcal{U} \cap \mathfrak{h}^\perp/T)$ is a smooth manifold. Indeed, $\tilde{J}^{-1}(0)/T = \tilde{J}^{-1}(0)/T'$ and we can check that the action of T' on $\tilde{J}^{-1}(0)$ is free. Let

$$(\mathfrak{u}_0)_{red} = \{f \in \mathcal{U} \cap \mathfrak{h}^\perp \mid f_{i,i+1} > 0, \quad i = 1, \dots, N-1\}. \quad (4.1.40)$$

Note that for $\xi \in \mathcal{U} \cap \mathfrak{h}^\perp$, there exists unique $h(\xi) \in T'$ such that $Ad_{h(\xi)}^{-1}\xi \in (\mathfrak{u}_0)_{red}$. We shall henceforth identify $\tilde{J}^{-1}(0)/T$ with $(U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times (\mathfrak{u}_0)_{red}$.

Now the dimension of the generic symplectic leaves of $(U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times (\mathfrak{u}_0)_{red}$ is given by

$$2N + \dim_{\mathbb{R}} \mathfrak{u}_0 - N - 2(N-1) = N(N-1) + 2. \quad (4.1.41)$$

Therefore, in order to show that the reduced Hamiltonian $\tilde{\mathcal{H}}_0$ is completely integrable in the sense of Liouville on these generic symplectic leaves, we have to exhibit $1 + \frac{N(N-1)}{2}$ functionally independent conserved quantities in involution. To do so, we define \tilde{L} by $\tilde{L}(q, p, f)(z) = Ad_{e^{-\frac{z}{12}q}} L(q, p, f)(z)$ for $(q, p, f) \in (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times (\mathfrak{u}_0)_{red}$. Then the characteristic polynomial of $\tilde{L}(q, p, f)(z)$ has the form

$$\det(\tilde{L}(q, p, f)(z) - w) = \sum_{r=0}^N \sum_{k=0}^r I_{rk}(q, p, f) c(z)^k w^{N-r}. \quad (4.1.42)$$

Now observe that $(\tilde{L}(q, p, f)(z))^* = \tilde{L}(q, p, f)(-\bar{z})$. Hence the functions $I_{r,2k}(q, p, f)$, $iI_{r,2k+1}(q, p, f)$ are real valued and provide the conserved quantities in involution. Clearly, among these are the $N-1$ Casimirs $I_{2k,2k}$ and $iI_{2k+1,2k+1}$, $k \geq 1$ (note that $I_{11} = 0$ on $(U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times (\mathfrak{u}_0)_{red}$). On the other hand, it follows from the explicit expression of \tilde{L} that

$$Ad_{e^{iq}} \tilde{L}(i\infty) = \tilde{L}(-i\infty). \quad (4.1.43)$$

Consequently, we obtain the relations

$$\sum_{k=0}^{\lfloor \frac{r-1}{2} \rfloor} (-1)^k I_{r,2k+1}(q, p, f) = 0 \quad (4.1.44)$$

for $r = 1, \dots, N$. Hence the total number of independent nontrivial integrals equals

$$1 + \sum_{r=2}^N (r-1) = 1 + \frac{1}{2}N(N-1), \quad (4.1.45)$$

as required.

4.2 Normal compact forms of some spin Calogero-Moser systems.

It is clear from (4.1.13) that R also induces a map $U \cap \mathfrak{h}_0 \longrightarrow L(L\mathfrak{g}_0, L\mathfrak{g}_0)$ which we will also denote by R .

Proposition 4.2.1. *The map $R : U \cap \mathfrak{h}_0 \longrightarrow L(L\mathfrak{g}_0, L\mathfrak{g}_0)$ is a solution of the mDYBE for the pair $(L\mathfrak{g}_0, \mathfrak{h}_0)$ with $c = -\frac{1}{4}$. Moreover, for $q \in U \cap \mathfrak{h}_0$, $\xi \in \mathfrak{g}_0$, we have $r_-^\#(q)\xi \in L\mathfrak{g}_0$.*

Proof. For $q \in U \cap \mathfrak{h}_0$, $X, Y \in L\mathfrak{g}_0$, the element $(dR(q)(\cdot)X, Y)$ must lie in \mathfrak{h}_0 because for $Z \in \mathfrak{h}_0$, $(dR(q)(Z)X, Y) \in \mathbb{R}$. The other assertion is clear from (4.1.11) as $\alpha(q) \in \mathbb{R}$ for $q \in U \cap \mathfrak{h}_0$. \square

We next introduce the trivial Lie groupoids

$$\Omega = (U \cap \mathfrak{h}_0) \times G_0 \times (U \cap \mathfrak{h}_0), \quad \Gamma = (U \cap \mathfrak{h}_0) \times LG_0 \times (U \cap \mathfrak{h}_0). \quad (4.2.1)$$

By Proposition 4.2.1 and (2.9), we can equip the dual bundle $A^*\Gamma \simeq (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times L\mathfrak{g}_0$ of $A\Gamma$ with the Lie algebroid structure associated to $R : U \cap \mathfrak{h}_0 \longrightarrow L(L\mathfrak{g}_0, L\mathfrak{g}_0)$. Hence its dual bundle $A\Gamma \simeq (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times L\mathfrak{g}_0$ has a Lie-Poisson structure. We shall also equip the dual bundle $A^*\Omega \simeq (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times \mathfrak{g}_0$ of the trivial Lie algebroid $A\Omega$ with the corresponding Lie-Poisson structure. The Poisson manifolds $A^*\Omega$ and $A\Gamma$ are Hamiltonian H_0 -spaces. Indeed, the actions are defined by expressions identical to (4.1.15) and (4.1.16) provided that we change $H^\mathbb{R}$ to H_0 and use the definitions of Ω and Γ in (4.2.1). On the other hand, the corresponding equivariant momentum maps are given by

$$J : A^*\Omega \longrightarrow \mathfrak{h}_0, \quad (q, p, \xi) \mapsto \Pi_{\mathfrak{h}_0}\xi \quad (4.2.2)$$

and

$$\gamma : A\Gamma \longrightarrow \mathfrak{h}_0, \quad (q, p, X) \mapsto p \quad (4.2.3)$$

where $\Pi_{\mathfrak{h}_0}$ is the projection map to \mathfrak{h}_0 relative to the decomposition $\mathfrak{g}_0 = \mathfrak{h}_0 \oplus (\mathfrak{h}_0)^\perp$.

As in Propositions 4.1.3 and 4.1.4, we have the following result in this case.

Proposition 4.2.2. *The map*

$$\rho : A^*\Omega \longrightarrow A\Gamma, \quad (q, -\Pi_{\mathfrak{h}_0}\xi, p + r_-^\#(q)\xi) \quad (4.2.4)$$

is an H_0 -equivariant Poisson map. Moreover, the map ρ gives a realization of the spin Calogero-Moser system on A^Ω with Hamiltonian*

$$\mathcal{H}(q, p, \xi) = \frac{1}{2} \sum_i p_i^2 - \frac{1}{8} \sum_{\alpha \in \Delta} \left(\frac{1}{\sin^2 \frac{1}{2}\alpha(q)} - \frac{1}{3} \right) \xi_\alpha \xi_{-\alpha} \quad (4.2.5)$$

in $A\Gamma$.

The Hamiltonian system in (4.2.5) will be called the *normal real form* of the complex holomorphic system $\mathcal{H}^{\mathbb{C}}$ in (4.1.24).

In the rest of the section, we shall reduce this normal real form to what we call the *normal compact form*. As the reader will see, the normal compact form has a natural family of conserved quantities in involution.

For this purpose, we introduce

$$\mathfrak{k}_0 = \sum_{\alpha \in \Delta} \mathbb{R}(e_{\alpha} - e_{-\alpha}), \quad \mathfrak{p}_0 = \mathfrak{h}_0 + \sum_{\alpha \in \Delta} \mathbb{R}(e_{\alpha} + e_{-\alpha}). \quad (4.2.6)$$

Then

$$\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0 \quad (4.2.7)$$

is a Cartan decomposition of \mathfrak{g}_0 . Let θ be the corresponding involution.

Proposition 4.2.3. *The map*

$$\kappa : A^*\Omega \longrightarrow A^*\Omega, \quad (q, p, \xi) \mapsto (q, p, \theta(\xi)) \quad (4.2.8)$$

is a Poisson involution with stable locus

$$A^*\Omega^{\kappa} = (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times \mathfrak{k}_0. \quad (4.2.9)$$

Proof. Since θ is a Cartan involution, it follows that \mathfrak{k}_0 and \mathfrak{p}_0 are orthogonal under $(\cdot, \cdot) |_{\mathfrak{g}_0 \times \mathfrak{g}_0}$. Using this property, we can show that $\theta^* = \theta$. Consequently, we conclude from Proposition 3.14 that κ is a Poisson involution. \square

As $\mathfrak{k}_0 = \mathfrak{u}_0 \cap \mathfrak{g}_0$, we shall call \mathfrak{k}_0 the *normal compact form* of \mathfrak{g} . Note, however, that \mathfrak{k}_0 is not at all a real form of \mathfrak{g} because its complexification is different from \mathfrak{g} . In view of this terminology, we define the *normal compact form* of $\mathcal{H}^{\mathbb{C}}$ to be the Hamiltonian system on $A^*\Omega^{\kappa}$ generated by

$$\begin{aligned} \tilde{\mathcal{H}}(q, p, \xi) &= (\mathcal{H} | A^*\Omega^{\kappa})(q, p, \xi) \\ &= \frac{1}{2} \sum_i p_i^2 + \frac{1}{8} \sum_{\alpha \in \Delta} \left(\frac{1}{\sin^2 \frac{1}{2}\alpha(q)} - \frac{1}{3} \right) |\xi_{\alpha}|^2. \end{aligned} \quad (4.2.10)$$

In order to discuss the realization of this Hamiltonian system, we introduce

$$s : L\mathfrak{g}_0 \longrightarrow L\mathfrak{g}_0, \quad s(X)(z) = - \sum_j \theta(X_j)(-z)^j. \quad (4.2.11)$$

Proposition 4.2.4. *The map*

$$\sigma : A\Gamma \longrightarrow A\Gamma, (q, p, X) \mapsto (q, -p, s(X)) \quad (4.2.12)$$

is a Poisson involution with stable locus

$$A\Gamma^\sigma = (U \cap \mathfrak{h}_0) \times \{0\} \times (L\mathfrak{g}_0)^s \quad (4.2.13)$$

where

$$(L\mathfrak{g}_0)^s = \{X \in L\mathfrak{g}_0 \mid X_{2j+1} \in \mathfrak{k}_0, X_{2j} \in \mathfrak{p}_0 \text{ for all } j\}. \quad (4.2.14)$$

Consequently, $Pr_3^*I((L\mathfrak{g}_0)^s)$ is a Poisson commuting family of functions on $A\Gamma^\sigma$.

Proof. Since $\theta = -1$ on \mathfrak{p}_0 and $\mathfrak{h}_0 \subset \mathfrak{p}_0$, we have $s_{\mathfrak{h}_0} = -\theta|_{\mathfrak{h}_0} = id_{\mathfrak{h}_0}$. Therefore, $s_{\mathfrak{h}_0}^* = id_{\mathfrak{h}_0}$. The rest of the proof of the first assertion is similar to the one for Proposition 4.1.6. On the other hand, the second assertion is just a consequence of (4.2.13) and Theorem 3.15. \square

Our next result shows that assumption A1 is satisfied. Its proof is similar to the one for Proposition 4.1.7.

Proposition 4.2.5. $\sigma \circ \rho = \rho \circ \kappa$.

From this proposition and (4.2.13), we see that the assumptions in Theorem 3.15 are satisfied. Hence we obtain the following result.

Theorem 4.2.6. (a) *The map*

$$\tilde{\rho} = \rho \mid A^*\Omega^\kappa : (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times \mathfrak{k}_0 \longrightarrow (U \cap \mathfrak{h}_0) \times \{0\} \times (L\mathfrak{g}_0)^s \quad (4.2.15)$$

is a Poisson map.

(b) $\tilde{\mathcal{H}} = L^*\tilde{Q}$ and admits $L^*I((L\mathfrak{g}_0)^s)$ as a family of conserved quantities in involution. Here, $L = Pr_3 \circ \tilde{\rho}$ and $\tilde{Q} = Q \mid (L\mathfrak{g}_0)^s$.

Remark 4.2.7. (a) The Hamiltonian system generated by $\tilde{\mathcal{H}}$ is completely integrable in the sense of Liouville on generic symplectic leaves of $A^*\Omega^\kappa$. This will also be treated in the forthcoming paper which we mentioned in the previous subsection.

(b) The normal compact forms of the rational and the elliptic spin calogero-Moser systems (corresponding to the dynamical r-matrices in Remark 4.1.12(b)) can also be treated in a similar fashion. Note that for the elliptic case, in order for the corresponding $R : U \longrightarrow L(L\mathfrak{g}, L\mathfrak{g})$ to induce a map $U \cap \mathfrak{h}_0 \longrightarrow L(L\mathfrak{g}_0, L\mathfrak{g}_0)$, we have to make an additional assumption, namely, we have to restrict to periods $2\omega_1$,

$2\omega_2$ (of the elliptic functions) for which the invariants $g_2 = 60 \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-4}$ and $g_3 = 140 \sum_{\omega \in \Lambda \setminus \{0\}} \omega^{-6}$ are real, where $\Lambda = 2\omega_1\mathbb{Z} + 2\omega_2\mathbb{Z}$.

As in Section 4.1, we shall close this subsection with a sketch of the Liouville integrability for $\mathfrak{g} = gl(N, \mathbb{C})$. In this case, U , \mathfrak{h}_0 and $r(q, z)$ are the same objects which appear at the end of section 4.1 and we have $\mathfrak{g}_0 = gl(N, \mathbb{R})$. Thus the factors \mathfrak{k}_0 and \mathfrak{p}_0 in the Cartan decomposition are respectively the set of skew-symmetric matrices and symmetric matrices in \mathfrak{g}_0 . Clearly the results above for simple Lie algebras have obvious analogs in this case. In particular, for the Hamiltonian system on $A^*\Omega^\kappa = (U \cap \mathfrak{h}_0) \times \mathfrak{h}_0 \times \mathfrak{k}_0$ generated by

$$\tilde{\mathcal{H}}(q, p, \xi) = \frac{1}{2} \sum_i p_i^2 + \frac{1}{8} \sum_{i \neq j} \left(\frac{1}{\sin^2(\frac{q_i - q_j}{2})} - \frac{1}{3} \right) |\xi_{ij}|^2, \quad (4.2.16)$$

its Hamiltonian equations of motion are given by the same expressions in (4.1.37) but with $\xi \in \mathfrak{k}_0$. In this case, the equations are associated with an orthogonal Floquet operator $F = e^{-i\lambda V} e^{-iH_0}$ and the Lax operator $L = Pr_3 \circ \tilde{\rho}$ on $A^*\Omega^\kappa$ has the same form as the one in (4.1.38) but with $\xi \in \mathfrak{k}_0$. Now the dimension of the generic symplectic leaves of $A^*\Omega^\kappa$ is given by $2N + \frac{N(N-1)}{2} - \left\lfloor \frac{N}{2} \right\rfloor = 2N + 2 \left\lfloor \frac{(N-1)^2}{4} \right\rfloor$. In order to show $\tilde{\mathcal{H}}$ is completely integrable in the sense of Liouville on these leaves, we have to exhibit $N + \left\lfloor \frac{(N-1)^2}{4} \right\rfloor$ functionally independent conserved quantities in involution. Put $\tilde{L}(q, p, \xi)(z) = Ad_{e^{-\frac{1}{12}q}} L(q, p, \xi)(z)$, then it is easy to check that $(\tilde{L}(q, p, \xi)(z))^T = \tilde{L}(q, p, \xi)(-z)$. Therefore the characteristic polynomial of $L(q, p, \xi)(z)$ is an even function of z . Hence we have

$$\det(\tilde{L}(q, p, \xi)(z) - w) = \sum_{r=0}^N \sum_{k=0}^{\left\lfloor \frac{r}{2} \right\rfloor} I_{rk}(q, p, \xi) c(z)^{2k} w^{N-r} \quad (4.2.17)$$

and the I_{rk} 's are conserved quantities in involution. Clearly, the functions $I_{2k,k}, k = 1, \dots, \left\lfloor \frac{N}{2} \right\rfloor$, are Casimirs. Therefore, the total number of nontrivial integrals is given by

$$\sum_{r=1}^N \left(\left\lfloor \frac{r}{2} \right\rfloor + 1 \right) - \left\lfloor \frac{N}{2} \right\rfloor = N + \left\lfloor \frac{(N-1)^2}{4} \right\rfloor. \quad (4.2.18)$$

5. Symmetric space spin Ruijsenaars-Schneider models and soliton dynamics of affine Toda field theory.

There is a well-known correspondence between the N -soliton solutions of the $A_n^{(1)}$ affine Toda field theory and some spin-generalized Ruijsenaars-Schneider equations [BH]. The goal of this section is to resolve a long-standing problem regarding the Hamiltonian formulation and the integrability of such equations.

Let $\mathfrak{g} = \mathfrak{gl}(N, \mathbb{C})$, and let \mathfrak{h} be the Cartan subalgebra of \mathfrak{g} consisting of diagonal matrices. We shall denote by $\mathfrak{g}^{\mathbb{R}}$ (resp. $\mathfrak{h}^{\mathbb{R}}$) the algebra \mathfrak{g} (resp. \mathfrak{h}) regarded as a real Lie algebra. It is well-known that

$$\mathfrak{u}(N) = \{ N \times N \text{ skew-Hermitian matrices} \} \quad (5.1)$$

is a compact real form of \mathfrak{g} . We shall denote by τ the conjugation of \mathfrak{g} with respect to $\mathfrak{u}(N)$. (Explicitly, $\tau(\xi) = -\xi^*$ for $\xi \in \mathfrak{g}$.) Clearly, the map

$$s = -\tau : \mathfrak{g}^{\mathbb{R}} \longrightarrow \mathfrak{g}^{\mathbb{R}} \quad (5.2)$$

is an involutive Lie algebra anti-morphism satisfying $s(\mathfrak{h}) = \mathfrak{h}$. In the following, the connected and simply-connected Lie groups which integrate $\mathfrak{g}^{\mathbb{R}}$, $\mathfrak{h}^{\mathbb{R}}$ will be denoted by $G^{\mathbb{R}}$ and $H^{\mathbb{R}}$ respectively. Then the Lie group anti-morphism $S : G^{\mathbb{R}} \longrightarrow G^{\mathbb{R}}$ corresponding to s is given by $S(g) = g^*$ for $g \in G^{\mathbb{R}}$.

In what follows, U will denote a fixed connected component of

$$\{q = \text{diag}(q_1, \dots, q_N) \in \mathfrak{h} \mid \sinh\left(\frac{1}{2}(q_i - q_j)\right) \neq 0 \text{ for all } i \text{ and } j\}.$$

We consider the solution $R : U \longrightarrow L(\mathfrak{g}, \mathfrak{g})$ of the mDYBE, given by

$$R(q)\xi = -\frac{1}{2} \sum_{i \neq j} \coth\left(\frac{1}{2}(q_i - q_j)\right) \xi_{ij} e_{ij}, \quad (5.3)$$

From this formula, it is clear that R induces a map $U \longrightarrow L(\mathfrak{g}^{\mathbb{R}}, \mathfrak{g}^{\mathbb{R}})$ which is a solution of the mDYBE for the pair $(\mathfrak{g}^{\mathbb{R}}, \mathfrak{h}^{\mathbb{R}})$. We shall denote this map also by R and from now onwards we shall only consider R as a map $U \longrightarrow L(\mathfrak{g}^{\mathbb{R}}, \mathfrak{g}^{\mathbb{R}})$. We now equip the trivial Lie groupoid

$$\Gamma = U \times G^{\mathbb{R}} \times U \quad (5.4)$$

with the coboundary dynamical Poisson structure associated to R . Since $H^{\mathbb{R}}$ is abelian, its action on Γ is given by

$$\mathcal{B} : H^{\mathbb{R}} \times \Gamma \longrightarrow \Gamma, \quad \mathcal{B}_h(u, g, v) = (u, hgh^{-1}, v). \quad (5.5)$$

Proposition 5.1. *The map*

$$\Sigma : (\Gamma, \{\cdot, \cdot\}_R) \longrightarrow (\Gamma, \{\cdot, \cdot\}_R), (u, g, v) \mapsto (\bar{v}, g^*, \bar{u}) \quad (5.6)$$

is a Poisson involution with stable locus

$$\Gamma^\Sigma = \{(u, g, \bar{u}) \in \Gamma \mid g = g^*\}. \quad (5.7)$$

Hence $(\Gamma, \{\cdot, \cdot\}_R, \Sigma)$ is a symmetric coboundary dynamical Poisson groupoid.

Proof. Using the pairing $(\xi, \eta)_\mathbb{R} = 2 \operatorname{Re} \operatorname{tr}(\xi \eta)$ on $\mathfrak{g}^\mathbb{R}$, it is straight forward to show that $s_{\mathfrak{h}^\mathbb{R}}^*(u) = s_{\mathfrak{h}^\mathbb{R}}(u) = \bar{u}$ for $u \in U$. By a similar calculation, we also have $s^* = s$. From this, it is easy to show that $s \circ R(q) \circ s^* = -R(\bar{q})$. Hence it follows from Proposition 3.12 that Σ is a Poisson involution. \square

Let T be the subgroup of $H^\mathbb{R}$ consisting of unitary diagonal matrices and let $\mathfrak{t} = \operatorname{Lie}(T)$.

Proposition 5.2. (a) *For all $d \in T$, $\mathcal{B}_d \circ \Sigma = \Sigma \circ \mathcal{B}_d$.*

(b) *$q - \bar{q} \in \mathfrak{t}$ for all $q \in U$.*

Proof. (a) From the definition of \mathcal{B} and Σ , we have $\mathcal{B}_d \circ \Sigma = \Sigma \circ \mathcal{B}_d$ for $d \in T$ iff $dg^*d^* = (dg^*d^*)^*$. But the latter is obvious.

(b) This assertion is clear. \square

Since we are dealing with the case in which the realization map is the identity map on Γ , it follows from Propositions 5.1 and 5.2 that assumptions G1-G4 in Section 3 are satisfied. Hence we have the following result by Proposition 3.19.

Corollary 5.3. *The stable locus Γ^Σ is a Hamiltonian T -space with equivariant momentum map $\tilde{\alpha} - \tilde{\beta} = \alpha - \beta \mid \Gamma^\Sigma$.*

Definition 5.4. The spin Ruijsenaars-Schneider models associated to R are the Hamiltonian systems on Γ generated by functions in $\mathbf{Pr}_2^*I(G^\mathbb{R})$. The symmetric space spin Ruijsenaars-Schneider models are the corresponding Hamiltonian systems on Γ^Σ generated by functions in $\mathbf{Pr}_2^*I((G^\mathbb{R})^S)$. Here we have used the same symbol \mathbf{Pr}_2 to denote the projection map from Γ to $G^\mathbb{R}$ and its restriction from Γ^Σ to $(G^\mathbb{R})^S$.

Note that in the case under consideration, we have $(\tilde{\alpha} - \tilde{\beta})^{-1}(0) \simeq U_s \times (G^\mathbb{R})^S$, where U_s consists of real diagonal matrices in U and $(G^\mathbb{R})^S$ consists of Hermitian

matrices in $G^{\mathbb{R}}$. As we are identifying $(\mathfrak{g}^{\mathbb{R}})^*$ with $\mathfrak{g}^{\mathbb{R}}$ using the pairing $(\cdot, \cdot)_{\mathbb{R}}$, it follows from (3.7) that the Poisson structure on Γ^{Σ} is given by

$$\{\tilde{\varphi}, \tilde{\psi}\}_{\Gamma^{\Sigma}}(u, g, \bar{u}) = -2(\iota\delta_1\tilde{\varphi}, D\tilde{\psi})_{\mathbb{R}} + 2(\iota\delta_1\tilde{\psi}, D\tilde{\varphi})_{\mathbb{R}} - 2(R(u)D\tilde{\varphi}, D\tilde{\psi})_{\mathbb{R}} \quad (5.8)$$

where

$$\delta_1\tilde{\varphi} := \frac{1}{2}(\delta_1\varphi + \overline{\delta_2\varphi}), \quad D\tilde{\varphi} := \frac{1}{2}(D\varphi + (D'\varphi)^*). \quad (5.9)$$

Hence we obtain the following result by applying Proposition 3.13.

Proposition 5.5. *Let $f(g) = 2\text{Re tr}(g)$, $g \in G^{\mathbb{R}}$ and let $F = Pr_2^*f$. Then the restriction of the Hamiltonian equations of motion generated by \tilde{F} to the invariant manifold $U_s \times (G^{\mathbb{R}})^S$ are given by*

$$\begin{aligned} \dot{q} &= \Pi_{\mathfrak{h}^{\mathbb{R}}} g, \\ \dot{g} &= g(R(q)g) - (R(q)g)g. \end{aligned}$$

In terms of the components of q and g , these read:

$$\ddot{q}_i = \dot{g}_{ii} = \frac{1}{2} \sum_{k \neq i} \coth((q_i - q_k)/2) g_{ik} g_{ki}, \quad (5.10a)$$

$$\begin{aligned} \dot{g}_{ij} &= \frac{1}{2} \coth((q_i - q_j)/2) g_{ij} (g_{jj} - g_{ii}) \\ &+ \frac{1}{2} \sum_{k \neq i, j} (\coth((q_i - q_k)/2) - \coth((q_k - q_j)/2)) g_{ik} g_{kj}, \quad i \neq j \end{aligned} \quad (5.10b)$$

Note that the equations in (5.10) for some special choice of g are exactly the ones derived by Braden and Hone in [BH] from the N -soliton solutions of the $A_n^{(1)}$ affine Toda field theory with purely imaginary coupling constant (cf. [KZ]). For the convenience of the reader, let us recall that the equations of motion of the $A_n^{(1)}$ affine Toda field theory (with imaginary coupling constant $i\beta$) are given by

$$\partial_+ \partial_- \phi_j + \frac{m^2}{2i\beta} \left(e^{i\beta(\phi_j - \phi_{j+1})} - e^{i\beta(\phi_{j-1} - \phi_j)} \right) = 0, \quad j = 0, \dots, n. \quad (5.11)$$

Here, ∂_{\pm} denotes differentiation with respect to the light-cone coordinates $x_{\pm} = \frac{1}{\sqrt{2}}(t \pm x)$ and the indices j are taken modulo $n+1$. In [BH], the authors were dealing with the solitonic sector of the theory, so they assumed in addition that $\sum_{j=0}^n \phi_j = 0$. Starting from the N -soliton solutions of (5.10) as derived by Hollowood [H]:

$$e^{i\beta\phi_j} = \frac{\tau_{j+1}}{\tau_j}, \quad (5.12)$$

where τ_j are the tau functions, Braden and Hone began by rewriting τ_j in determinantal form. As it turned out, they found that

$$\tau_j = \det \left(1 + e^{ij\Theta/2} V e^{ij\Theta/2} \right) \quad (5.13)$$

where V is an invertible skew-Hermitian matrix depending on x_{\pm} (and the $2N$ soliton parameters) satisfying

$$\dot{V} = \frac{1}{2}(\Lambda V + V \Lambda), \quad \cdot = \partial_{\pm} \quad (5.14)$$

In (5.13) and (5.14), $\Theta = \text{diag}(\theta_1, \dots, \theta_N)$ and

$$\Lambda = \text{diag} \left(\pm \sqrt{2}m \exp(\mp \eta_1) \sin(\theta_1/2), \dots, \pm \sqrt{2}m \exp(\mp \eta_N) \sin(\theta_N/2) \right) \quad (5.15)$$

where θ_j are discrete parameters associated with the solitons taking values in $\{\frac{2\pi k}{n+1} \mid k = 1, \dots, n\}$, and η_j are the rapidities. As V is skew-Hermitian and invertible, there exists a unitary matrix U (unique up to transformations $U \rightarrow \delta U$, where $\delta \in T$) which diagonalizes V , i.e.

$$ie^q = UVU^* \quad (5.16)$$

where q is real diagonal. Using U , define an invertible Hermitian matrix g by:

$$g = U \Lambda U^*. \quad (5.17)$$

Then under the evolution as defined by (5.14), the authors in [BH] showed that q and g satisfy (5.10)! Of course, in this context, the variable g depends on the choice of U . Clearly, the system which is independent of such a choice is the corresponding reduced system on the Poisson quotient $U_s \times ((G^{\mathbb{R}})^S/T)$. As a consequence of Theorem 3.20, we therefore conclude that the reduced system has N Poisson commuting integrals.

Remark 5.7. There are several questions which we have not addressed in our discussion above. One of these has to do with the nature of the transformation between the $2N$ soliton parameters and the dynamical variables in $U_s \times ((G^{\mathbb{R}})^S/T)$. On the other hand, there should be corresponding results for the soliton solutions associated with the other affine Lie algebras in [OTU]. Of course, there is also the question of Liouville integrability. We hope to return to these questions in future work.

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